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# Extremal Theory for Spectrum of Random Discrete Schrödinger Operator. I. Asymptotic Expansion Formulas

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**Abstract** We consider the spectral problem for the random Schrödinger operator on the multidimensional lattice torus increasing to the whole of lattice, with an i.i.d. potential (Anderson Hamiltonian). We obtain the explicit almost sure asymptotic expansion formulas for the extreme eigenvalues and eigenfunctions in the intermediate rank case, provided the upper distributional tails of potential decay at infinity slower than the double exponential function. For the fractional-exponential tails (including Weibull's and Gaussian distributions), extremal type limit theorems for eigenvalues are proved, and the strong influence of parameters of the model on a specification of normalizing constants is described. In the proof we use the finite-rank perturbation arguments based on the cluster expansion for resolvents.

The results of our paper illustrate a close connection between extreme value theory for spectrum and extremal properties of i.i.d. potential. On the other hand, localization properties of the corresponding eigenfunctions give an essential information on long-time intermittency for the parabolic Anderson model.

**Keywords** Anderson Hamiltonian · Random potential · Extreme eigenvalues · Principal eigenvalue · Extremal type limit theorem · Rare scatterers model

## 1 Introduction

The Anderson model is given by the Hamiltonian

$$\mathcal{H} = \kappa\Delta + \xi(\cdot)$$

acting on $l^2(\mathbb{Z}^\nu)$ ("the space of wave functions"). Here $\Delta$ is the lattice Laplacian, i.e., $\Delta\psi(x) := \sum_{|y-x|=1} \psi(y)$ $(x \in \mathbb{Z}^\nu)$, $\kappa > 0$ is a diffusion constant, and the potential $\xi(\cdot) = \xi^{(\omega)}(\cdot)$ consists of independent identically distributed (i.i.d.) random variables $\xi(x)$ $(x \in \mathbb{Z}^\nu)$

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with a common distribution function $F(\cdot)$. In the multidimensional case ($\nu \geqslant 1$), many authors (e.g., [1, 2, 12, 17]) have shown that for any $\kappa > 0$, there exists a (nonrandom) constant $L = L(\kappa, F(\cdot)) \in \mathbb{R}$ such that the spectrum in $(L, \infty)$ is purely point, i.e.,

$$\mathrm{Spect}(\mathcal{H}) \cap (L, \infty) = \mathrm{Spect}_{pp}(\mathcal{H}) \cap (L, \infty) \neq \varnothing,$$

and the corresponding eigenfunctions decay exponentially with probability 1, provided $F(\cdot)$ is Hölder continuous and $\xi(0)$ has some finite statistical moments. (For small $\kappa$, the whole spectrum $\mathrm{Spect}(\mathcal{H})$ is purely point.) This phenomenon is referred to as Anderson localization for disordered systems; see [13, 15, 35, 39] for a detailed survey on the subject.

If $V \subset \mathbb{Z}^\nu$ is the $\nu$-dimensional torus of the volume $|V|$, the Anderson model in $l^2(V)$ is given by the Hamiltonian (finite matrix):

$$\mathcal{H}_V = \kappa \Delta_V + \xi(\cdot)$$

with periodic boundary data. Clearly the spectrum of $\mathcal{H}_V$ is a discrete and finite set, say, $\mathrm{Spect}(\mathcal{H}_V) = \{\lambda_{k,V} :\ 1 \leqslant k \leqslant |V|\}$, where

$$\lambda_{1,V} \geqslant \lambda_{2,V} \geqslant \cdots \geqslant \lambda_{|V|,V}.$$

For $\kappa = 0$, this variational series becomes $\xi_{1,V} \geqslant \xi_{2,V} \geqslant \cdots \geqslant \xi_{|V|,V}$.

In our paper, letting $V$ increases to $\mathbb{Z}^\nu$ (i.e., $|V| \to \infty$), we study the asymptotic structure (in particular, extremal type limit theorems) of extremely high energy spectrum

$$\mathrm{Spect}(\mathcal{H}_V) \cap (L_{V,\varepsilon}, \infty)$$

and localization properties of the corresponding eigenfunctions. Here constants $L_{V,\varepsilon}$ (high levels) are chosen to be $L_{V,\varepsilon} := f((1-\varepsilon)\log|V|)$, where $f(\cdot)$ stands for the inverse function of $-\log(1 - F(\cdot))$ and $0 < \varepsilon < 1/2$. Throughout the paper the distribution function $F(\cdot)$ is assumed to satisfy the following conditions:

$$\lim_{t\to\infty} (f(t) - f(\delta t)) = \infty \quad \text{for any } 0 < \delta < 1 \tag{1.1}$$

and

$$\begin{aligned} &(F(t+s) - F(t-s))\,|\log s|^\mu = \mathrm{O}(1) \\ &\text{as } t \to \infty \text{ and } s \downarrow 0 \text{ simultaneously} \end{aligned} \tag{1.2}$$

for some $\mu > \mu_0(\varepsilon)$. A function $F(\cdot)$ satisfying (1.2) is called log-Hölder continuous of order $\mu$ at infinity. Clearly the class of distribution functions satisfying (1.1) and (1.2) includes Weibull's distributions

$$1 - F(t) = \mathrm{e}^{-t^\alpha} \quad (t \geqslant 0) \tag{1.3}$$

with arbitrary $\alpha > 0$, and those with fractional-double exponential tails

$$1 - F(t) = \mathrm{const}\exp\left\{-\mathrm{e}^{t^\gamma}\right\} \quad (t \geqslant 0) \tag{1.4}$$

for $0 < \gamma < 1$.

Weibull's distributions play an important role in the physical theory of "Lifshitz tails" because of the existence of bifurcations with respect to $\alpha$; see Chap. 2 in [34]. The task here is to find the exact asymptotic expansion formulas for the upper tails of the (limiting) spectral distribution function, the structure of which is expected to depend strongly on the parameter $\alpha$; see Sect. 2.4 of our paper.

We stress the fact that the analysis of the boundary part spectrum is essential to understand the long-time intermittent behavior for the parabolic problems associated with the Anderson Hamiltonian via spectral representation of solutions. In [11, 20, 22–24, 30], the authors have studied various aspects of asymptotic intermittency for the parabolic Anderson model, in particular, the almost sure asymptotic behavior of solutions and, as a by-product, the almost sure asymptotic formulas for the principal (i.e., largest) eigenvalue $\lambda_{1,V}$ of the Hamiltonian $\mathcal{H}_V$. They have pointed out the crucial role of the fractional-double exponential distribution tails (1.4) because of the strong influence of both the distributional parameter $\gamma$ and the diffusion constant $\kappa$ on the asymptotic behavior of the model; see also Sect. 2 of our paper.

The main idea of the present study of spectrum comes from the mathematical theory of "rare scatterers" and is based on the cluster expansion method for resolvents. The latter leads to the explicit expansion formulas for extreme eigenvalues and eigenfunctions of the Hamiltonian $\mathcal{H}_V$. This method was particularly used by Golitsyna and Molchanov [26] to analyze the spectral problems for Hamiltonians on the whole of $\mathbb{Z}^\nu$ with an infinite sequence of (widely spaced) random potential peaks. The main feature of the subject is that the interaction between potential peaks can be neglected and the eigenpairs associated with a block of peaks can be determined by the eigenpairs associated with the separate peaks. We notice that the rare scatterers model is "typical" for the Anderson Hamiltonian $\mathcal{H}_V$ on finite regions $V$ under certain conditions on $F(\cdot)$. Let $\mathcal{G}_V^{(z)}(\lambda;\cdot,\cdot)$ be Green's function of the Hamiltonian $\kappa\Delta_V + (1-\delta_z)\xi(\cdot)$ on $V$, where $\{z\} \subset V$ are locations of exceedances $\xi(z) \geqslant L_{V,\varepsilon}$. We show that with probability 1 the extreme eigenvalue $\lambda = \lambda(z)$ of $\mathcal{H}_V$, associated with $z$, is a solution to the dispersion equation $\mathcal{G}_V^{(z)}(\lambda; z, z) = 1/\xi(z)$ and the corresponding eigenfunction is $\mathcal{G}_V^{(z)}(\lambda;\cdot,z)$. By finite-rank perturbation arguments this dispersion equation is approximated by the dispersion equation for the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot) + \xi(z)\delta_z$, where $\widetilde{\xi}(\cdot)$ is a "noise" potential. The asymptotic analysis of extreme eigenvalues of $\mathcal{H}_V$ is therefore reduced to the investigation of the principal eigenvalues of the "single peak" Hamiltonians, which in turn are expanded in a certain series over $\widetilde{\xi}(x)/\xi(z)$ $(x \in V)$.

The origins of our paper are the announcements in [5, 9]. In the case of Weibull's distribution with $\alpha < 2$, extremal type limit theorems for eigenvalues of the Anderson Hamiltonian $\mathcal{H}_V$ (as $V \uparrow \mathbb{Z}^\nu$) were earlier proved by Grenkova *et al.* [29] by combining Gerzhgorin's theorem and the minimum-maximum principle for eigenvalues. Grenkova *et al.* [28] studied limit distributions of extreme eigenvalues of the one-dimensional random operators by developing the phase formalism (which, however, is specific to dimension one only). Gärtner and Molchanov [24] applied the variational principle to study the first two asymptotic terms of the principal eigenvalue of the Anderson Hamiltonian $\mathcal{H}_V$ (as $V \uparrow \mathbb{Z}^\nu$) under the general conditions on $F(\cdot)$ extending (1.1); see also [11, 30]. For the case of spatially continuous Schrödinger operator with Poisson obstacle potential, we refer to [37, 44].

Recently, there is much progress toward the mathematical treatment of the boundary part spectrum of random matrices, as the matrix volume increases. Limit theorems for the first few extreme eigenvalues were proved, e.g., in [16, 42, 45, 46] for Wigner random matrices, and in [10, 31, 43] for sample covariance random matrices; see also [36] for the general theory of random matrices.

The results of our paper illustrate a close connection between the extremal properties of random potential and the asymptotic structure of the boundary part spectrum. We refer the reader to Sect. 2.2 for a detailed discussion on this relationship.

We can think of the asymptotic results for spectrum of the Anderson Hamiltonian $\mathcal{H}_V$ as a natural extension of extremal type limit theorems for i.i.d. random fields $\xi(\cdot)$ in $V \uparrow \mathbb{Z}^\nu$ (the latter corresponds to the case of $\kappa = 0$ in our model). Extreme value theory for i.i.d. random sequences and fields is a well-developed branch of the probability theory; see, e.g., [19, 33, 40]. We also point out that the results presented here for the i.i.d. potential could be extended by using our methods to other classes of homogeneous ergodic potentials including Gaussian fields with correlated values [4], moving average fields, Markov chains, etc. Further extensions of our results include the case of Schrödinger operators on graphs $\mathfrak{G}$ when each $x \in \mathfrak{G}$ has a fixed number of neighbors (for example, Bethe lattice) as well as the case when the Laplacian $\Delta$ is replaced by a translation invariant finite range matrix operator $\mathcal{T}\psi(x) = \sum_{y\in\mathbb{Z}^\nu} T(y)\psi(x-y)$, where $T(\cdot)$ is a real (nonrandom) function with a finite support.

The organization of the paper is as follows:

Section 2 is, in fact, a continuation of Introduction. We first discuss the main results of the present paper on the asymptotic structure (as $|V| \to \infty$) of the spectrum $\mathrm{Spect}(\mathcal{H}_V) \cap (L_{V,\varepsilon}, \infty)$ (see Sect. 2.1). We then treat their connections with extremal properties of the potential $\xi(\cdot)$ in $V$ (Sect. 2.2) as well as connections with the almost sure long-time behavior of the solutions to the parabolic problem associated with $\mathcal{H}$ (Sect. 2.3). At the end, some remarks about the asymptotics for the upper tails of the spectral distribution function are carried out (Sect. 2.4).

In Sect. 3, we investigate the almost sure extremal properties of the i.i.d. potential $\xi(\cdot)$ in torus $V$ increasing to $\mathbb{Z}^\nu$. In particular, we study the asymptotic behavior of exceedances of high levels $L_{V,\varepsilon}$.

In Sect. 4, we treat the almost sure asymptotic structure of the extreme eigenvalues $\lambda_{k,V}$ in the intermediate rank case $1 \leqslant k = \mathrm{O}(|V|^\varepsilon)$, provided $F(\cdot)$ satisfies conditions (1.1) and (1.2). At the end, we briefly discuss some extensions of the previous results to the case when $F(\cdot)$ has the double exponential distribution tails and the diffusion constant $\kappa$ is small.

Section 5 provides extremal type limit theorems for a finite number of the first extreme eigenvalues $\lambda_{k,V}$, i.e., fixed rank case. In Sect. 5.1, we consider the class of distribution functions $F(\cdot)$ satisfying (1.1) and (1.2). In Sect. 5.2, we briefly discuss the class of distribution functions $F(\cdot)$ satisfying $-\log(1-F(t)) = \mathrm{o}(t^3)$ as $t \to \infty$, i.e., a potential with extremely sharp peaks. In this case, condition (1.2) on log-Hölder continuity is removed.

In Sect. 6, the results of Sects. 4 and 5 are extended to a special class of distributions with fractional-exponential tails (containing Weibull's distributions (1.3) for arbitrary $\alpha > 0$ as well as Gaussian distributions). Bifurcations with respect to distributional parameters are studied.

Appendices A and B form the "deterministic" part of the paper. In Appendix B, we study the spectral problem for rare scatterers model in $V \subset \mathbb{Z}^\nu$ (or on the whole of $\mathbb{Z}^\nu$). In order to derive the explicit estimates for eigenpairs we use finite-rank perturbation arguments based on the expansion of resolvents over paths (cluster expansion, expansion over $\kappa\Delta$) given in Appendix A. Appendices A and B present self-contained topics of general theory of Schrödinger operator, and may therefore be considered of independent interest.

Our forthcoming papers [7, 8] are devoted to a generalization and extension of the results of the present article. In [7], we study joint limit distributions of a finite number of the first extreme eigenvalues $\lambda_{k,V}$ of the Hamiltonian $\mathcal{H}_V$ (as $V \uparrow \mathbb{Z}^\nu$), provided $-\log(1 - F(t)) = \mathrm{o}(t^3)$ as $t \to \infty$ (i.e., heavy distributional tails). In this case, potential $\xi(\cdot)$ in $V$ possesses extremely pronounced peaks and, therefore, the eigenvalue $\lambda_{k,V}$ is asymptotically close to the $k$th extreme value $\xi_{k,V}$ of $\xi(\cdot)$ in $V$; here $k$ is fixed (see Sect. 5.2 for discussion). In [8], we study localization properties of the eigenfunctions corresponding to the first $K$

extreme eigenvalues of $\mathcal{H}_V$, provided $F(\cdot)$ satisfies the conditions of the present paper or the conditions of [7]. Bifurcations with respect to distributional parameters are studied.

Notation. Representation of i.i.d. potential $\xi(\cdot)$

By $\mathbb{Z}^\nu$ we denote the $\nu$-dimensional integer lattice, and by $\mathbb{R}_+$ the positive half-axis, and $\mathbb{N} := \{1, 2, \ldots\}$. Given $n \in \mathbb{N}$, we introduce the periodic norm $|\cdot|$ on $\mathbb{Z}^\nu$ by

$$|x| := \min_{y \in 2n\mathbb{Z}^\nu} \left(|x^1 - y^1| + \cdots + |x^\nu - y^\nu|\right)$$

for $x = (x^1, \ldots, x^\nu) \in \mathbb{Z}^\nu$, and let $V$ be the $\nu$-dimensional torus obtained by identifying opposite faces of the cube in $\mathbb{Z}^\nu$, centred at the origin of $\mathbb{Z}^\nu$ with sides of the length $2n + 1$ parallel to the lattice axes.

By $\mathcal{G}_V(\lambda; x, y)$ $(x \in V,\ y \in V)$ we denote Green's function of the Hamiltonian $\mathcal{H}_V$ in $l^2(V)$, viz. $\mathcal{G}_V(\lambda; x, y) := \mathcal{G}_V(\lambda)\delta_y(x) := (\lambda - \mathcal{H}_V)^{-1}\delta_y(x)$. Here $\delta_y(\cdot)$ stands for the Kronecker symbol, i.e., $\delta_y(x) := 1$ if $x = y$, and $\delta_y(x) := 0$ if $x \neq y$. Let $\{\psi(x; \lambda) :\ x \in V\}$ be an eigenfunction associated with $\lambda \in \operatorname{Spect}(\mathcal{H}_V)$ and normalized by the condition $\sum_{x \in V} \psi(x; \lambda)^2 = 1$.

Let $\log_j$ denote the $j$times iterated logarithm. For real $a$ and $b$, we write $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. Given a subset $U \subset \mathbb{Z}^\nu$, we write $|U|$ for the number of its elements. Let $\mathbb{1}\{E\}$ or $\mathbb{1}_E$ stand for the indicator function of a subset $E \subset \mathbb{R}$, $E \subset \mathbb{N}$, etc. The summation over $x \in V :\ a \leqslant |x| \leqslant b$ is abbreviated to $\sum_{x:\ a \leqslant |x| \leqslant b}$ or simply to $\sum_{a \leqslant |x| \leqslant b}$. For positive numbers $a_N$ and $b_N$ $(N \in \mathbb{N})$ we write $a_N \asymp b_N$ as $N \to \infty$ if and only if $0 < \liminf_N a_N/b_N \leqslant \limsup_N a_N/b_N < \infty$. By $t_0$, $|V_0|$, etc. we denote various large numbers, values of which may change from one appearance to the next. Similarly, const, const$'$, etc. stand for various positive constants.

Throughout the paper we suppose that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{E}$ stand for the expectation with respect to $\mathbb{P}$. Given a sequence of random variables $X_N = X_N^{(\omega)}$ $(N \in \mathbb{N};\ \omega \in \Omega)$ and positive numbers $a_N$ $(N \in \mathbb{N})$, we write $X_N = \mathrm{o}(a_N)$ as $N \to \infty$ (resp., $X_N = \mathrm{O}(a_N)$ as $N \to \infty$) in probability if $\lim_N \mathbb{P}(|X_N| > a_N\varepsilon) = 0$ for each $\varepsilon > 0$ (resp., $\lim_{\varepsilon \downarrow 0} \limsup_N \mathbb{P}(|X_N| > a_N/\varepsilon) = 0$).

The following *representation of an i.i.d. field* will play the key role in a number of our statements. Given a right-continuous distribution function $F(t)$ $(t \in \mathbb{R})$, let us introduce the function

$$f(s) := \inf\left\{t :\ 1 - F(t) \leqslant \mathrm{e}^{-s}\right\} \quad (0 < s < \infty). \tag{1.5}$$

Note that $f(\cdot)$ is left-continuous and

$$1 - F(f(s)) \leqslant \mathrm{e}^{-s} \leqslant 1 - F(f(s)-) \quad \text{for each } 0 < s < \infty; \tag{1.6}$$

see pp. 5–7 in [41]. Let $\eta(x)$ $(x \in \mathbb{Z}^\nu)$ be independent exponentially distributed random variables with mean 1. Clearly the random variables

$$\xi(x) := f(\eta(x)) \quad (x \in \mathbb{Z}^\nu) \tag{1.7}$$

are independent and have a (common) distribution function $F(\cdot)$; see p. 3 in [41]. Given a sample $\eta(\cdot)$ in $V$, we associate the sites $z_{k,V} \in V$ $(1 \leqslant k \leqslant |V|)$ with the variational series

$$\eta(z_{1,V}) > \eta(z_{2,V}) > \cdots > \eta(z_{|V|,V}); \tag{1.8}$$

the inequalities are strict with probability 1 due to the continuity of exponential distribution. According to (1.7) and (1.8) the variables

$$\xi_{1,V} := \xi(z_{1,V}) \geqslant \xi_{2,V} := \xi(z_{2,V}) \geqslant \cdots \geqslant \xi_{|V|,V} := \xi(z_{|V|,V}) \tag{1.9}$$

form the variational series based on the sample $\xi(\cdot)$ in $V$. For any $0 < \varepsilon < 1$, the first $|V|^{\varepsilon}$ larger values $\xi_{k,V}$ (resp., $z_{k,V}$) are referred to as $\xi_V$-peaks (resp., locations of $\xi_V$-peaks).

## 2 General Remarks on Extremal Theory for Spectrum and Related Topics

### 2.1 Asymptotic Results for the Upper Part of Spectrum

The main results of the present paper can be briefly rephrased as follows:

1) **Localization properties.** Fix $0 < \varepsilon < 1/2$, and assume that the distribution function $F(\cdot)$ of the i.i.d. potential satisfies conditions (1.1) and (1.2) with $\mu = \mu(\varepsilon) > 0$ large enough. Write $L_{V,\varepsilon} := f((1-\varepsilon)\log|V|)$, and let $z_{k,V} \in V$ $(1 \leqslant k \leqslant |V|)$ be locations of extreme values of $\xi(\cdot)$ in $V$. Then the eigenfunctions $\psi(\cdot;\lambda_k)$ corresponding to $\lambda_k \in \mathrm{Spect}(\mathcal{H}_V) \cap (L_{V,\varepsilon}, \infty)$ are exponentially well localized, i.e., with probability 1

$$|\psi(x;\lambda_k)| \leqslant \exp\{-A(L_{V,\varepsilon})|x - z_{\tau(k),V}|\} \quad (x \in V)$$

for some (random) $\tau(k) = \tau_V(k) \in \{1, 2, \dots, |V|\}$ and for some constants $A(L_{V,\varepsilon}) \to \infty$ as $|V| \to \infty$ (Theorem 4.1).

In fact, the sites $\{z_{\tau(k),V}\colon\ \lambda_k \geqslant L_{V,\varepsilon}\} \subset V$ form (as $|V| \to \infty$) extremely rare set of locations of $\xi_V$-peaks. In particular case where $F(\cdot)$ is Weibull's distribution (1.3) and $K \in \mathbb{N}$ is fixed, we show that for $\alpha < 3$, $\tau_V(K) \to K$; whereas in the case of $\alpha > 3$, $\tau_V(K) \to \infty$ and $\log \tau_V(K) = \mathrm{o}(\log|V|)$ (as $|V| \to \infty$) in probability (Theorems 4.1 and 6.3, and Remark 6.5 below; see also [8]).

2) **Extremal type limit theorems.** Assume that $F(\cdot)$ is Weibull's distribution and $K \in \mathbb{N}$ is fixed. Then there exist constants $B_V = B_V(\alpha, \kappa, \nu) \in \mathbb{R}$ such that the normalized spectral interval lengths

$$\frac{\lambda_{1,V} - \lambda_{2,V}}{\alpha^{-1}(\log|V|)^{1/\alpha-1}}, \dots, \frac{\lambda_{K-1,V} - \lambda_{K,V}}{\alpha^{-1}(\log|V|)^{1/\alpha-1}}, \frac{\lambda_{K,V} - B_V}{\alpha^{-1}(\log|V|)^{1/\alpha-1}}$$

are asymptotically (as $|V| \to \infty$) mutually independent and have limiting joint distributions with the density

$$\exp\left\{-t_1 - \cdots - (K-1)t_{K-1} - Kt_K - \mathrm{e}^{-t_K}\right\}$$

for all $t_k \geqslant 0$ $(1 \leqslant k \leqslant K-1)$ and all $t_K \in \mathbb{R}$. This convergence result for $\xi_{k,V}$ replacing $\lambda_{k,V}$ $(1 \leqslant k \leqslant |V|)$ obviously holds with $b_V := f(\log|V|) = (\log|V|)^{1/\alpha}$ replacing $B_V$. In the case of $\alpha < 2$, one has that $B_V = b_V$; whereas for $\alpha \geqslant 2$, the constants $B_V$ and $b_V$ are different. For instance, $B_V = b_V + C_0 b_V^{-1}$ if $2 \leqslant \alpha < 3$, $B_V = b_V + C_0 b_V^{-1} + C_1 b_V^{-(\alpha+1)/(\alpha-1)} + (C_2 \log b_V + C_3) b_V^{1-\alpha}$ if $3 \leqslant \alpha < (3 + \sqrt{17})/2$ and so on (Theorems 6.2 and 6.3, and Corollary 6.4). Asymptotic equations for the normalizing constants $B_V$ are derived as well (Theorem 6.3). By this limit theorem with $\alpha > 1$, the eigenvalue $\lambda_{K,V}$ is expanded in the series (deterministic flow) $B_V = b_V + C_0 b_V^{-1} + C_1 b_V^{-(\alpha+1)/(\alpha-1)} + \cdots$ plus small random fluctuations of the order $\mathrm{O}(b_V^{1-\alpha})$ as $|V| \to \infty$.

Gärtner and Molchanov [24] have obtained (by using the variational principle) the second-order expansion formula for the principal eigenvalue $\lambda_{1,V}$ (as $V \uparrow \mathbb{Z}^\nu$) under the following condition on $F(\cdot)$:

$$\lim_{t\to\infty}(f(t) - f(\delta t)) = -\rho \log \delta \quad \text{for each } 0 < \delta < 1, \tag{2.1}$$

for some $0 \leqslant \rho \leqslant \infty$. The latter naturally extends condition (1.1) (i.e., the case $\rho = \infty$). If $0 < \rho < \infty$, the class of distribution functions (2.1) includes the double exponential case

$$1 - F(t) = \exp\{-\mathrm{e}^{t/\rho}\} \quad (t \in \mathbb{R}). \tag{2.2}$$

If $\rho = 0$, the typical examples of (2.1) are $F(\cdot)$ satisfying (1.4) with $\gamma > 1$ and $F(\cdot)$ satisfying $F(t^0) = 1$ for some $t^0 \in \mathbb{R}$ (i.e., bounded from above potentials).

**Proposition 2.1** (Sect. 2.4 in [24]) *Assume that $F(\cdot)$ is a continuous function satisfying condition (2.1) for some $0 \leqslant \rho \leqslant \infty$, and the additional condition $f(t + C\log t) - f(t) \to 0$ as $t \to \infty$, for some $C > 1$. Then with probability* 1

$$\lambda_{1,V} = f(\log|V|) + 2\nu\kappa q(\rho/\kappa) + \mathrm{o}(1) \quad \text{as } |V| \to \infty. \tag{2.3}$$

*Here the nonrandom function $q(\cdot)$ may be expressed in terms of a variational problem, having the following properties: $q(\cdot)$ is nonincreasing and convex, $0 < q(\cdot) < 1$ in $(0,\infty)$; $q(0) = 1$ and $q(\infty) = 0$.*

*Moreover, according to Corollary 4.5 of our paper, we have that*

$$q(\rho) = (2\rho\log\rho)^{-1}(1 + \mathrm{o}(1)) \quad \text{as } \rho \to \infty.$$

We point out that the first term in (2.3) is equal (with o(1) accuracy) to $\xi_{1,V} =$ the maximum of $\xi(\cdot)$ in $V$ (see Remark 3.3 below). The second term in (2.3) contains information about localization properties of the eigenfunction $\psi(\cdot\,;\lambda_{1,V})$. Using the maximum-minimum principle for eigenvalues, we obtain that, for any realization $\xi(\cdot)$,

$$\xi_{1,V} \leqslant \lambda_{1,V} \leqslant \xi_{1,V} + 2\nu\kappa \tag{2.4}$$

for all $V$. In view of (2.4), limit (2.3) tells us that the principal eigenvalue $\lambda_{1,V}$ achieves as $|V| \to \infty$ its lower (resp., upper) bound, provided $F(\cdot)$ satisfies (2.1) with $\rho = \infty$ (resp., $\rho = 0$). We emphasize the case of (2.1) when the ratio $\rho/\kappa$ is large enough (Theorem 4.4). The latter indicates the intermediate situation between asymptotically complete localization (if $\rho = \infty$) and noncomplete localization (if $0 \leqslant \rho < \infty$) for the eigenfunction $\psi(\cdot\,;\lambda_{1,V})$.

Of course, limit (2.3) for $\rho = 0$ is trivial; see the simple arguments of the proof of (2.3), $\rho = 0$, at the end of Sect. 2.2. In the case of $\rho = 0$, Biscup and König [11], Hofstad *et al.* [30] have recently obtained more accurate almost sure asymptotic formulas for the eigenvalue $\lambda_{1,V}$ under mild regularity conditions on the upper distributional tails of $\xi(0)$ in terms of the cumulant generating function $\log \mathbb{E}\,\mathrm{e}^{t\xi(0)}$ as $t \to \infty$. These asymptotic formulas for $\lambda_{1,V}$ consist of deterministic terms reflecting, in particular, localization properties of the eigenfunction $\psi(\cdot\,;\lambda_{1,V})$. The main accent of the proofs here is that $\lambda_{1,V}$ is dominated by the principal eigenvalue of the Hamiltonian $\mathcal{H}_U$ on the random region $U = U_V \subset V$ with the following a.s. asymptotic properties. First, $U$ unboundedly increases, but the size of $U$ is much smaller than that of $V$, i.e., $|U| = \mathrm{o}(|V|)$ as $V \uparrow \mathbb{Z}^\nu$ and, second, the potential on $U$ attains extremely large values and is of particular preferred shape. This shape (called "optimal") is specified by a deterministic variational formula.

### 2.2 Relation to Extremal Properties of i.i.d. Potential

We now discuss the relationship between the asymptotic results on the upper part of spectrum, $\mathrm{Spect}(\mathcal{H}_V) \cap (L_{V,\varepsilon}, \infty)$, and the extremal properties of the i.i.d. potential $\xi(\cdot)$ in $V \uparrow \mathbb{Z}^\nu$.

Assume that $F(\cdot)$ satisfies condition (1.1) (i.e., the upper distributional tails are thicker than the double exponential) and the additional condition (1.2) of log-Hölder continuity, provided $\mu > \mu_0(\theta)$ and $0 < \theta < 1/2$. Then extreme values of a "typical" sample $\xi(\cdot)$ in $V$ possess a strongly pronounced geometric structure which can be described as follows:

For $0 < \varepsilon < \theta$, let the subset $\Pi_{V,\varepsilon} \subset V$ (resp., $\Pi_{V,\theta} \subset V$) consists of sites at which $\xi_V$-exceedances of the level $L_{V,\varepsilon}$ (resp., $L_{V,\theta}$) occur. Let $\widetilde{\xi}(x) := \xi(x)$ if $x \in V \backslash \Pi_{V,\theta}$ and be zero otherwise (a "noise" random potential). Then one can find constants $c_1 > c_2 > 0$ and $C > 2\nu\kappa$ such that with probability 1

$$\frac{1}{2}|V|^{\varepsilon} \leqslant |\Pi_{V,\varepsilon}| \leqslant |\Pi_{V,\theta}| \leqslant 2|V|^{\theta}, \tag{2.5}$$

$$\min\left\{|x-y|:\ x \in \Pi_{V,\theta},\ y \in \Pi_{V,\theta},\ x \neq y\right\} \geqslant |V|^{c_1}, \tag{2.6}$$

$$\min\left\{|\xi(x)-\xi(y)|:\ x \in \Pi_{V,\theta},\ y \in \Pi_{V,\theta},\ x \neq y\right\} \geqslant \exp\{-|V|^{c_2}\} \tag{2.7}$$

and

$$\min\left\{\xi(x)-\widetilde{\xi}(y):\ x \in \Pi_{V,\varepsilon},\ y \in V\right\} > C \tag{2.8}$$

for each sufficiently large $V$ (Lemmas 3.1 and 3.5).

From the physical point of view, properties (2.5)–(2.8) mean that there is no resonance between $\xi_V$-peaks in the Anderson model for large $V$. Thus, finite-rank perturbation arguments of Appendices A and B show that an eigenvalue $\lambda(z^0)$ of $\mathcal{H}_V$ associated with a site $z^0 \in \Pi_{V,\varepsilon}$ is approximately close to the principal eigenvalue $\widetilde{\lambda}(z^0)$ of the "single-peak" Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot) + \xi(z^0)\delta_{z^0}$ in $l^2(V)$, viz.

$$|\lambda(z^0) - \widetilde{\lambda}(z^0)| \leqslant \exp\{-|V|^{c_1}\} \tag{2.9}$$

for each sufficiently large $V$ (Theorem B.3 in Appendix B). On the other hand, the eigenvalue $\lambda = \widetilde{\lambda}(z^0)$ is the maximal solution to the equation

$$\widetilde{\mathcal{G}}_V(\lambda; z^0, z^0) = \frac{1}{\xi(z^0)}, \tag{2.10}$$

where $\widetilde{\mathcal{G}}_V(\lambda; x, y)$ $(x \in V,\ y \in V)$ is Green's function of the Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot)$ in $l^2(V)$ (Remark B.5). Moreover, $\widetilde{\mathcal{G}}_V(\lambda; z^0, z^0)$ is expanded over $\kappa\Delta$

$$\widetilde{\mathcal{G}}_V(\lambda; z^0, z^0) = \sum_{\Gamma} \kappa^{|\Gamma|} \prod_{v \in V} \left(\lambda - \widetilde{\xi}(v)\right)^{-n_v(\Gamma)} \tag{2.11}$$

(Lemma A.2 in Appendix A), where the sum $\sum_\Gamma$ is taken over all paths $\Gamma : v_0 := z^0 \to v_1 \to \cdots \to v_m := z^0$ in $V$ such that $|v_i - v_{i-1}| = 1$ for each $1 \leqslant i \leqslant m$ and each $m \in \mathbb{N}$, $n_v(\Gamma)$ denotes the number of times the path $\Gamma$ visits the site $v \in V$, $|\Gamma| := \sum_{v \in V} n_v(\Gamma) - 1 \geqslant 0$. By (2.10) and (2.11) we expand $\widetilde{\lambda}(z^0)$ over $\widetilde{\xi}(x)/\xi(z^0)$ $(x \in V)$ to represent $\widetilde{\lambda}(z^0)$ in the form

$$\widetilde{\lambda}(z^0) = \xi(z^0) + \frac{2\nu\kappa^2}{\xi(z^0)} + \kappa^2 \sum_{l=1}^{3} \sum_{|x-z^0|=1} \frac{\widetilde{\xi}(x)^l}{\xi(z^0)^{l+1}} + \mathrm{O}\left(\sum_{|x-z^0|=1} \frac{\widetilde{\xi}(x)^4}{\xi(z^0)^5}\right)$$

$$+\,\mathrm{O}\Bigg(\sum_{\substack{|x-z^0|=1\\|y-z^0|=1\\|z-z^0|\le 2}}\frac{1}{(\xi(z^0)-\widetilde{\xi}(x))(\xi(z^0)-\widetilde{\xi}(y))(\xi(z^0)-\widetilde{\xi}(z))}\Bigg) \tag{2.12}$$

as $|V|\to\infty$; $z^0\in\Pi_{V,\varepsilon}$. We finally note that

$$\min\big\{|\widetilde{\lambda}(x)-\widetilde{\lambda}(y)|:\ x\in\Pi_{V,\varepsilon},\ y\in\Pi_{V,\varepsilon},\ x\neq y\big\}\geqslant\exp\{-|V|^{c_2}\} \tag{2.13}$$

(cf. (2.7)). From (2.9), (2.12) and (2.13) we obtain that, with probability 1, any $k$th extreme eigenvalue $\lambda_{k,V}$ of $\mathcal{H}_V$ is approximately close to the $k$th extreme value of the random series (2.12) for each $1\leqslant k\leqslant\frac{1}{2}|V|^{\varepsilon}$, for each large $V$.

For fixed $K\in\mathbb{N}$, the asymptotic behavior of the first $K$ extreme values among $\widetilde{\lambda}(x)$ ($x\in\Pi_{V,\varepsilon}$) depends strongly on the decay rate of the gaps (spacings) $\xi_{K,V}-\xi_{K+1,V}$ as $|V|\to\infty$, or equivalently, on the decay rate of the upper distributional tails at infinity. The following two examples illustrate this phenomenon:

1) Assume that $F(\cdot)$ is Weibull's distribution and $\alpha<3$, so that $\xi(\cdot)$ satisfies (2.5)–(2.8) and, in addition,

$$\xi_{K+1,V}^2(\xi_{K,V}-\xi_{K+1,V})\to\infty$$

in probability. Then the third and all other terms in expansion (2.12) with $z^0=z_{K,V}$ are asymptotically smaller than $\xi_{K,V}-\xi_{K+1,V}$. This in turn implies that, as $|V|\to\infty$, the eigenfunction $\psi(\cdot\,;\lambda_{K,V})$ is completely localized on $z_{K,V}$, so that $\lambda_{K,V}$ corresponds to $z_{K,V}$, viz. $\lambda_{K,V}\leftrightarrow z_{K,V}$ (Theorems 6.2 and 6.3 when $\alpha<3$).

2) In the case of Weibull's distribution with $\alpha>3$, properties (2.5)–(2.8) again hold true and, in addition,

$$\xi_{K+1,V}^2(\xi_{K,V}-\xi_{K+1,V})\to 0$$

in probability. Then the third term in (2.12) becomes asymptotically essential. Therefore, the correspondence $\lambda_{K,V}\leftrightarrow z_{K,V}$ fails and the eigenfunction $\psi(\cdot\,;\lambda_{K,V})$ is completely localized on $z_{\tau(K),V}$ with $\tau(K)$ different from $K$ in limit as $|V|\to\infty$ (Theorems 6.2 and 6.3 when $\alpha>3$, and Remark 6.5; see also [8] for a more detailed discussion on localization properties).

Assume now that $F(\cdot)$ satisfies (2.1) with $0\leqslant\rho<\infty$, i.e., the upper distributional tails are like double exponential or thinner than those. Then qualitative new type bifurcations with respect to both the distributional parameter and the diffusion constant are observed. To illustrate this phenomenon, let us consider the following two cases:

1) Let (2.1) be fulfilled with $\rho>0$, but the ratio $\rho/\kappa$ being large. (The latter means, in particular, that either the degree of disorder of the potential is high, or the diffusion is weak.) Assume, in addition, that $F(\cdot)$ is log-Hölder continuous at infinity with $\mu>\mu_0(\theta)$. Then properties (2.5)–(2.8) of $\xi(\cdot)$ are still valid, and hence the asymptotic structure of $\mathrm{Spect}(\mathcal{H}_V)\cap(L_{V,\varepsilon},\infty)$ is quite similar to the case of $\rho=\infty$. In particular, the principal eigenvalue $\lambda_{1,V}$ corresponds to the isolated site $z_{\tau(1),V}\in V$ such that

$$\frac{\log\tau(1)}{\log|V|}\to\varepsilon^*(\rho/\kappa)\in(0,1/2)\quad\text{as } |V|\to\infty$$

(Theorem 4.4 and Corollary 4.5).

2) Assume now that (2.1) is fulfilled with $\rho = 0$. Then $\xi_V$-peaks are asymptotically too close to each other in height, viz.

$$\xi_{[|V|^\delta],V} - \xi_{[|V|^\varepsilon],V} \to 0 \quad \text{as } |V| \to \infty$$

with probability 1, for all constants $0 \leqslant \delta < \varepsilon < 1$ (Lemma 3.1(i)); i.e., property (2.8) does not hold. In this case, therefore, the principal eigenvalue does not longer correspond to an isolated potential peak, but to an extremely large "island" of peaks of comparable amplitude [11, 30]. In order to stress this correspondence, let us check that with probability one $\lambda_{1,V} - f(\log |V|) \to 2\nu\kappa$ (as $|V| \to \infty$) by applying the following transparent arguments.

Given $i \in \mathbb{N}$, $d = d(i) \in \mathbb{N}$ and sufficiently large $V$, let $V^{(i)}$ denote the collection of "boxes" $U(x) := \{y\colon\ |y - x| \leqslant d\} \subset V$ of the volume $|U(x)| = i$ and $\operatorname{diam} U(x) = d$, $x \in V$. By Lemma 3.4 below, with probability 1 there is a "box" $U^{(i)} \in V^{(i)}$ such that $\min_{x\in U^{(i)}} \xi(x) \geqslant L_{V,\varepsilon}$ for some $\varepsilon \in (\frac{i-1}{i}; \frac{i}{i+1})$ and each $V \supset V_0(w; i)$. Abbreviate $\xi^{(i)}(x) := \xi(x)$ if $x \in U^{(i)}$, and $\xi^{(i)}(x) := -\infty$ otherwise. From (2.1) with $\rho = 0$ we note that almost sure $\xi^{(i)}(z) = f(\log |V|) + \mathrm{o}(1)$ as $|V| \to \infty$, for all $z \in U^{(i)}$. Therefore, by the minimum-maximum principle, we have with probability one that $\lambda_{1,V} \geqslant \lambda^{(i)} + f(\log |V|) + \mathrm{o}(1)$ as $|V| \to \infty$, where $\lambda^{(i)}$ is the principal eigenvalue of $\kappa\Delta$ in $l^2(U^{(i)})$ which tends to $2\nu\kappa$ as $U^{(i)}$ unboundedly increases. In view of the upper bound in (2.4), the above arguments show that the main contribution to the eigenvalue $\lambda_{1,V}$ comes from an "island" of $\xi_V$-peaks $\xi(z) = f(\log |V|) + \mathrm{o}(1)$, $z \in U^{(i)}$, with $|U^{(i)}| = i \to \infty$. This yields the desired limit for $\lambda_{1,V}$.

As mentioned by Gärtner and Molchanov [24] in the case of (2.1) for some $0 < \rho < \infty$, the eigenvalue $\lambda_{1,V}$ corresponds to an "island" of $\xi_V$-peaks with asymptotically finite support $U^{(i)} \subset V$, $|U^{(i)}| = i$, for some $i \in \mathbb{N}$. Notice that, given $i$, the subsets $U^{(i)}$ are located asymptotically far away from each other; cf. Lemma 3.4 of our paper.

## 2.3 Relation to the Parabolic Problems

The asymptotic results for the spectrum, Spect$(\mathcal{H}_V)$, give the essential information on the long-time intermittent behavior of the solution to the parabolic equation

$$\frac{\partial u(s,x)}{\partial s} = \kappa \sum_{|y|=1} (u(s,x+y) - u(s,x)) + \xi(x)u(s,x), \quad s \in \mathbb{R}_+,\ x \in \mathbb{Z}^\nu, \tag{2.14}$$

with the homogeneous initial datum $u(0,\cdot) \equiv 1$. Equation (2.14) is to describe an evolution of a particle system of the branching type with random birth and death rates (random medium). Thus, given a realization $\xi(\cdot)$, the nonnegative solution $u(s,x)$ is the expected number of particles at time $s$ at site $x$, where expectation is taken over branching mechanism and diffusion, but not over $\xi(\cdot)$ [14, 21]. Under mild conditions on the i.i.d. field $\xi(\cdot)$ (the existence of statistical moments of $\xi(0)$), (2.14) has, with probability 1, an unique nonnegative solution which admits the Feynman–Kac representation; see Sect. 2 in [23]. Note that, for each $s \geqslant 0$, the solution $u(s,\cdot)$ is a homogeneous ergodic random field.

The notion of intermittency refers to an appearance (as $s \to \infty$) of the lattice regions (i.e., "islands") which are far away from each other and provide the essential contribution to the solutions of (2.14). This phenomenon has been explained at the physical level of rigor by Zel'dovich *et al.* [48]. Gärtner and Molchanov [23, 24] have presented a rigorous definition of intermittency and derivation of the second-order expansion formulas for statistical moments and almost sure behavior of $u(s,x)$ as $s \to \infty$, for fixed $x \in \mathbb{Z}^\nu$. For the upper distributional tails of $\xi(0)$ thinner than the double exponential, Biscup and König [11], Hofstad *et al.* [30] have obtained more accurate expansion formulas for $u(s,x)$. A spatial correlation structure of $u(s,\cdot)$ $(s\to\infty)$ have been investigated by Gärtner and Hollander [20], and the geometric picture of intermittency by Gärtner *et al.* [22]. In [20, 22, 24] the emphasize has been made on the double exponential distribution tails (2.2). The latter indicates the critical situation between formation of (widely spaced) single peaks of $u(s,\cdot)$ in the case of (2.1) with $\rho=\infty$ and formation of (widely spaced) extremely large flat "islands" of peaks in behavior of $u(s,\cdot)$ in the case of (2.1) with $\rho=0$ [11, 30].

Let us sketch the derivation of asymptotic formulas for the almost sure behavior (as $s\to\infty$) of the solution $u(s,0)$ by using spectral representation for $u(\cdot,\cdot)$. We follow the arguments of Sect. 2 in [24]. Assume that $F(\cdot)$ satisfies condition (2.1). We claim the condition $f(t+C\log t)-f(t)\to 0$ as $t\to\infty$, for some $C>1$. The latter excludes the class of random potentials the maxima of which have "sharp" random fluctuations (cf. Remark 3.3). Assume, in addition, that $F(\cdot)$ is continuous, $F(t)<1$ for all $t\in\mathbb{R}$ and, in dimension $\nu=1$, $\int_{-\infty}^{-1}\log|t|\,\mathrm{d}F(t)<\infty$. Write $V(s):=\mathbb{Z}^\nu\cap[-s(\log s)^{1+\varepsilon},s(\log s)^{1+\varepsilon}]^\nu$ for small $\varepsilon>0$, and let $u_{V(s)}(\cdot,\cdot)$ be a solution to the corresponding equation in $V(s)$ with periodic boundary condition. We then obtain with probability 1 that

$$u(s,0)=u_{V(s)}(s,0)+\mathrm{o}(1)\quad\text{as } s\to\infty,$$

by the standard cut-off procedure for $u(s,0)$ exploiting the fact that the main asymptotic contribution to the Feynman–Kac representation of $u(s,0)$ is given by particle trajectories which stay in box $V(s)$ during the whole time interval $[0,s]$. On the other hand, the solution $u_{V(s)}(\cdot,\cdot)$ admits the spectral representation

$$u_{V(s)}(s,x)=\sum_{k=1}^{|V(s)|}\exp\{\lambda_{k,V(s)}s-2\nu\kappa s\}\big(\psi(\cdot;\lambda_{k,V(s)}),1\big)_{V(s)}\psi(x;\lambda_{k,V(s)});$$

here, remember, $\lambda_{k,V(s)}$ and $\psi(\cdot;\lambda_{k,V(s)})$ stand for the $k$th eigenvalue and the corresponding eigenfunction of the Hamiltonian $\mathcal{H}_{V(s)}$ (the eigenfunctions are chosen to form an orthonormal basis of $l^2(V(s))$); $(\cdot,\cdot)_V$ stands for the inner product in $l^2(V)$ and 1 denotes the function taking everywhere value 1. This implies that with probability 1

$$u_{V(s)}(s,0)=\exp\big\{\lambda_{1,V(s)}s-2\nu\kappa s+\mathrm{o}(s)\big\}.$$

The latter and Proposition 2.1 imply the following statement.

**Proposition 2.2** (Sect. 2.1 in [24]) *Under the above conditions on $F(\cdot)$ we have with probability* 1 *that*

$$\frac{\log u(s,0)}{s}=f(\nu\log s)-2\nu\kappa(1-q(\rho/\kappa))+\mathrm{o}(1)\quad\text{as } s\to\infty;\tag{2.15}$$

*here $q(\cdot)$ is specified in Proposition* 2.1.

Recall that the first term on the right-hand side of (2.15) is equal (with accuracy o(1)) to the maximum of the potential in $V(s)$ and the second term describes the shape of the potential in the neighborhood of its maximum.

### 2.4 Asymptotics for the Spectral Distribution Function

Let us define the limiting spectral distribution function of random Schrödinger operator $\mathcal{H} = \kappa\Delta + \xi(\cdot)$ on $l^2(\mathbb{Z}^\nu)$, provided $\xi(\cdot)$ is a homogeneous ergodic field. Given torus $V \subset \mathbb{Z}^\nu$, by $N_V(t)$ we denote the empirical spectral distribution function, viz.

$$N_V(t) := \frac{1}{|V|} \sum_k \mathbb{1}\{\lambda_{k,V} \leqslant t\} \quad (t \in \mathbb{R}),$$

where, as before, $\lambda_{k,V} \in \mathrm{Spect}(\mathcal{H}_V)$ $(1 \leqslant k \leqslant |V|)$. The properties of homogeneity and ergodicity of $\xi(\cdot)$ are sufficient for $N_V(t)$ to converge with probability 1, as $|V| \to \infty$, to the nonrandom limit $N(t)$ at all points $t \in \mathbb{R}$; here $N(\cdot)$ is a continuous distribution function such that $N(-\infty) = 0$ and $N(\infty) = 1$, i.e., the so-called integrated density of states. See, e.g., Chap. 2 in [39].

We point out the fact that the support of the spectral distribution function $N(\cdot)$ coincides, with probability 1, with the spectrum of the Hamiltonian $\mathcal{H}$. (Note that in the case of i.i.d. potential, $\mathrm{Spect}(\mathcal{H})$ in turn coincides with the algebraic sum of the close interval $[-2\nu\kappa, 2\nu\kappa]$ and the support of distribution function $F(\cdot)$ of $\xi(0)$; see, e.g., Chap. 2 in [39].) The further task is to investigate the asymptotic behavior of the tails $1 - N(t)$ at the upper edge of spectrum, i.e., the so-called Lifshitz tails. We first mention the following fundamental estimates for $N(t)$:

$$F(t - 2\nu\kappa) \leqslant N(t) \leqslant F(t + 2\nu\kappa) \quad (t \in \mathbb{R}). \tag{2.16}$$

The proof of this is trivial. Indeed, since the norm of the Laplacian $\kappa\Delta$ in $l^2(V)$ does not exceed $2\nu\kappa$, each eigenvalue $\lambda_{k,V}$ is bounded from above (resp., from below) by the $k$th eigenvalue of the diagonal operator $\xi(\cdot) + 2\nu\kappa$ in $l^2(V)$ (resp., $\xi(\cdot) - 2\nu\kappa$), i.e. $\xi_{k,V} - 2\nu\kappa \leqslant \lambda_{k,V} \leqslant \xi_{k,V} + 2\nu\kappa$ for all $1 \leqslant k \leqslant |V|$ and all $V$. Therefore

$$F_V(t - 2\nu\kappa) \leqslant N_V(t) \leqslant F_V(t + 2\nu\kappa) \quad (t \in \mathbb{R}),$$

where $F_V(\cdot)$ denotes the empirical distribution function of the potential values $\xi(x)$ $(x \in V)$. Passing to the limit as $|V| \to \infty$, from the individual ergodic theorem we obtain the claimed assertion (2.16).

The bounds (2.16) for $N(\cdot)$ immediately imply the following asymptotic formulas for the tails $1 - N(t)$.

**Proposition 2.3** *Write* $F^*(t) := -\log(1 - F(t))$, *and assume that* $F^*(t) < \infty$ *for each* $t \in \mathbb{R}$ (*i.e., unbounded from above potentials*). *If*

$$\lim_{t\to\infty} \frac{F^*(t + C)}{F^*(t)} = 1 \quad \textit{for all } C > 0, \tag{2.17}$$

*then*

$$\lim_{t\to\infty} \frac{\log(1 - N(t))}{\log(1 - F(t))} = 1.$$

*If*

$$\lim_{t\to\infty} \big(F^*(t + C) - F^*(t)\big) = 0 \quad \textit{for all } C > 0, \tag{2.18}$$

*then*

$$\lim_{t\to\infty}\frac{1-N(t)}{1-F(t)}=1.$$

Note that condition (2.17) is fulfilled for Weibull's distribution (1.3) with $\alpha>0$ and for the distributions of form (1.4) with $0<\gamma<1$ (in fact, condition (2.17) follows from (1.1)). Condition (2.18) (which includes the case (1.3) with $0<\alpha<1$) guarantees the exact asymptotics for $1-N(t)$. In the case of $\alpha\geqslant 1$, the asymptotics of $1-N(t)$ is distinguished by the existence of bifurcations with respect to $\alpha$. However, a derivation of the exact expansion formulas for $1-N(t)$ when $\alpha\geqslant 1$ still remains an unsolved problem. In view of extremal type limit theorems for eigenvalues (cf. Sect. 2.1 above), we can guess the first few terms of the expansion when $\alpha\geqslant 1$: $1-N(t)=\exp\{-t^{\alpha}+C^{0}t^{\alpha-2}(1+\mathrm{o}(1))\}$ as $t\to\infty$.

If $F(\cdot)$ satisfies (2.1) with $0<\rho<\infty$ (i.e., the double exponential case), the asymptotic logarithmic formulas for the Lifshitz tails are given in [25]. In [11], these formulas are extended to the case of bounded from above potentials, i.e., $\operatorname{esssup}\xi(0)<\infty$. The proof of their results relies on the asymptotic formulas for statistical moments of the solutions of the corresponding parabolic problem. For a detailed background of the theory of the spectral distribution function and related topics, the reader is referred to [13, 34, 39, 47].

## 3 Extremal Properties of i.i.d. Potential

In this section we investigate the asymptotic geometric structure of extreme values of $\{\xi(x):\ x\in V\}$ as $V\uparrow\mathbb{Z}^{\nu}$. We assume throughout that $\xi(\cdot)$ is an i.i.d. random field with a (common) right-continuous distribution function $F(t)$ such that $F(t)<1$ for each $t\in\mathbb{R}$ (i.e., unbounded from above potential).

### 3.1 Sets of Exceedances

We first study the almost sure (a.s.) asymptotic properties of the set of exceedances

$$\Pi_{V,\varepsilon}:=\{x\in V:\ \xi(x)\geqslant L_{V,\varepsilon}\},\tag{3.1}$$

where

$$L_{V,\varepsilon}:=f((1-\varepsilon)\log|V|)$$

for $0<\varepsilon<1$; here $f(\cdot)$ is given by (1.5). We are especially interested in the minimum distance between two exceedances

$$r(\Pi_{V,\varepsilon}):=\begin{cases}\min\{|x-y|:\ x\in\Pi_{V,\varepsilon},\ y\in\Pi_{V,\varepsilon},\ x\neq y\} & \text{if } |\Pi_{V,\varepsilon}|\geqslant 2,\\ |V|^{1/\nu} & \text{if } |\Pi_{V,\varepsilon}|\leqslant 1;\end{cases}\tag{3.2}$$

here the second line is to involve the case of the "trivial" rare subset $\Pi_{V,\varepsilon}$.

Clearly, for each $0<\varepsilon<1$,

$$\begin{aligned}&\mathbb{E}|\Pi_{V,\varepsilon}|\quad(=\text{the mean number of exceedances})\\&\quad=|V|(1-F(L_{V,\varepsilon}-))\geqslant|V|^{\varepsilon}\to\infty\quad\text{as }|V|\to\infty,\end{aligned}$$

according to the upper estimate in (1.6). We insert the following condition

$$\frac{1-F(t-)}{1-F(t)} \to 1 \quad \text{as } t \to \infty, \tag{3.3}$$

which together with (1.6) guarantees the convergence

$$\lim_V \frac{\mathbb{E}|\Pi_{V,\varepsilon}|}{|V|^{\varepsilon}} = 1.$$

**Lemma 3.1** [3, 6] (i) *If $F(\cdot)$ satisfies condition (3.3), then a.s.*

$$\lim_V \frac{|\Pi_{V,\varepsilon}|}{|V|^{\varepsilon}} = 1 \quad \text{for } 0 < \varepsilon < 1$$

*and*

$$\lim_V \frac{\log r(\Pi_{V,\varepsilon})}{\log |V|} = \frac{1-2\varepsilon}{\nu} \quad \text{for } 0 < \varepsilon < \frac{1}{2}.$$

(ii) *If $\varepsilon > \frac{1}{2}$, then a.s.*

$$\lim_V r(\Pi_{V,\varepsilon}) = 1.$$

In what follows, we need some additional statements related to the statements of Lemma 3.1.

*Remark 3.2* From the proof of Lemma 3.1 in [3, p. 273] we know that

(i)

$$\max\left\{\mathbb{P}\left(|\Pi_{V,\varepsilon}| \geqslant 2|V|^{\varepsilon}\right), \mathbb{P}\left(|\Pi_{V,\varepsilon}| \leqslant \frac{1}{2}|V|^{\varepsilon}\right)\right\} \leqslant \mathrm{e}^{-\mathrm{const}|V|^{\varepsilon}} \tag{3.4}$$

for any $0 < \varepsilon < 1$, any $V$, and some $\mathrm{const} = \mathrm{const}(\varepsilon, \nu) > 0$;

(ii)

$$\mathbb{P}\left(r(\Pi_{V,\varepsilon}) \leqslant |V|^{(1-\varepsilon-\theta)/\nu}\right) \leqslant |V|^{-\mathrm{const}'}$$

for any $0 < \varepsilon < \theta < 1/2$ and for some $\mathrm{const}' = \mathrm{const}'(\varepsilon, \theta, \nu) > 0$.

*Remark 3.3* Fix $K \in \mathbb{N}$. By Theorems 1 and 2 of [41, pp. 407, 408] we have that $\limsup_V |\eta_{K,V} - \log|V||/\log_2|V| = 1/K$ a.s. Combining this with formulas (1.7)–(1.9), we obtain the following assertion: if

$$f(t + C\log t) - f(t) \to 0 \quad \text{as } t \to \infty$$

for some $C > 1/K$, then a.s.

$$\xi_{K,V} = f(\log|V|) + \mathrm{o}(1) \quad \text{as } |V| \to \infty.$$

The latter means that the $K$th larger values $\xi_{K,V}$ are close to extremely high levels $L_{0,V} := f(\log|V|)$ (cf. Lemma 3.1).

In the next lemma, we treat the asymptotic structure of subsets of bounded size formed by high-level exceedances by $\xi(\cdot)$ in $V$, i.e., statement (ii) of the previous lemma is specified more precisely.

Given $i \in \mathbb{N}$ and $R \in \mathbb{N}$, let $V^{i,R}$ stand for a family of subsets $U \subset V$ such that $|U| = i$ and $\operatorname{diam} U := \max_{x \in U, y \in U} |x - y| \leqslant R$. Write $\xi(U) := \min_{x \in U} \xi(x)$ and

$$\Pi_{V,\varepsilon}^{i,R} := \{U \in V^{i,R} :\ \xi(U) \geqslant L_{V,\varepsilon}\}.$$

As in (3.2) we abbreviate

$$r(\Pi_{V,\varepsilon}^{i,R}) := \min\{\operatorname{dist}(U, U') :\ U \in \Pi_{V,\varepsilon}^{i,R},\ U' \in \Pi_{V,\varepsilon}^{i,R},\ U \neq U'\} \quad \text{if } |\Pi_{V,\varepsilon}^{i,R}| \geqslant 2,$$

and $r(\Pi_{V,\varepsilon}^{i,R}) := |V|^{1/\nu}$ if $|\Pi_{V,\varepsilon}^{i,R}| \leqslant 1$, by convention; here $\operatorname{dist}(U, U')$ stands for the lattice distance between subsets $U, U' \subset V$.

**Lemma 3.4** [6] *Fix constants $i \in \mathbb{N}\backslash\{1\}$ and $R \in \mathbb{N}$ arbitrarily. If $F(\cdot)$ satisfies condition (3.3), then the following assertions hold true almost sure.*

*In the case of $\frac{i-1}{i} < \varepsilon < \frac{i}{i+1}$,*

$$\lim_V \frac{\log |\Pi_{V,\varepsilon}^{i,R}|}{\log |V|} = 1 + i(\varepsilon - 1)$$

*and*

$$\liminf_V \frac{\log r(\Pi_{V,\varepsilon}^{i,R})}{\log |V|} \geqslant \frac{2i(1-\varepsilon) - 1}{\nu};$$

*meanwhile, for $\varepsilon < \frac{i-1}{i}$,*

$$\lim_V |\Pi_{V,\varepsilon}^{i,R}| = 0.$$

## 3.2 Differences in Height

We now study the asymptotic behavior of the minimum of the gaps $\xi_{k,V} - \xi_{l,V}$ for $1 \leqslant k < l = \mathrm{O}(|V|^{\varepsilon})$. Write

$$s(\Pi_{V,\varepsilon}) := \begin{cases} \min\{|\xi(x) - \xi(y)| :\ x \in \Pi_{V,\varepsilon}, y \in \Pi_{V,\varepsilon},\ x \neq y\} & \text{if } |\Pi_{V,\varepsilon}| \geqslant 2, \\ 0 & \text{if } |\Pi_{V,\varepsilon}| \leqslant 1; \end{cases} \tag{3.5}$$

here $\Pi_{V,\varepsilon} \subset V$ is defined by (3.1) and $0 < \varepsilon < 1$.

**Lemma 3.5** *Fix $0 < \varepsilon < 1$ arbitrarily, and let $s(\Pi_{V,\varepsilon})$ be defined by (3.5). If $F(\cdot)$ is log-Hölder continuous of order $\mu > 0$ at infinity (i.e., (1.2) holds), then a.s.*

$$\limsup_V \frac{\log\{-\log(s(\Pi_{V,\varepsilon}) \wedge 1)\}}{\log |V|} \leqslant \frac{1+\varepsilon}{\mu}. \tag{3.6}$$

*Proof* It suffices to show (3.6) for a sequence of torus $V(l)$ $(l \in \mathbb{N})$ with the following properties:

$$V(l) \text{ monotone increases and } |V(l)| = 2^l(1 + \mathrm{o}(1)) \text{ as } l \to \infty. \tag{3.7}$$

Abbreviate $L_V := L_{V,\varepsilon}$, $\Pi_V := \Pi_{V,\varepsilon}$, and $d_V := \exp\{-|V|^d\}$, where $d > (1+\varepsilon)/\mu$. By (3.4) we have that

$$\begin{aligned}
&\mathbb{P}\,(s(\Pi_V) < d_V) \\
&\quad \leqslant \mathbb{P}\left(s(\Pi_V) < d_V, \frac{1}{2}|V|^\varepsilon < |\Pi_V| < 2|V|^\varepsilon\right) + \mathrm{O}\left(\mathrm{e}^{-\mathrm{const}|V|^\varepsilon}\right) \\
&\quad \leqslant \sum_{V'} \mathbb{P}\left(s\,(\Pi_V) < d_V, \Pi_V = V'\right) + \mathrm{O}\left(\mathrm{e}^{-\mathrm{const}|V|^\varepsilon}\right),
\end{aligned} \tag{3.8}$$

where the summation $\sum_{V'}$ is taken over all subsets $V' \subset V$ such that $\frac{1}{2}|V|^\varepsilon < |V'| < 2|V|^\varepsilon$. Clearly

$$\begin{aligned}
&\mathbb{P}\left(s(V') < d_V, \Pi_V = V'\right) \\
&\quad \leqslant \sum_{\substack{x\in V', y\in V' \\ x\neq y}} \mathbb{P}\,(|\xi(x) - \xi(y)| < d_V, \xi(x) \geqslant L_V, \xi(y) \geqslant L_V) \\
&\qquad \times \mathbb{P}\,(\xi(0) \geqslant L_V)^{|V'|-2}\,\mathbb{P}\,(\xi(0) < L_V)^{|V\setminus V'|} \\
&\quad \leqslant \mathrm{const}\,|V|^2 \mathbb{P}\,(|\xi(0) - \xi(y)| < d_V, \xi(0) \geqslant L_V)\,\mathbb{P}\left(\Pi_V = V'\right)
\end{aligned} \tag{3.9}$$

for fixed $y \in \mathbb{Z}^\nu \setminus \{0\}$. Combining (3.8) and (3.9), we get that, for any $V \supset V_0$,

$$\begin{aligned}
&\mathbb{P}\,(s(\Pi_V) < d_V) \\
&\quad \leqslant \mathrm{const}\,|V|^2 \mathbb{E}\,[(F\,(\xi(0) + d_V) - F\,(\xi(0) - d_V))\,\mathbb{1}\,\{\omega\colon\ \xi(0) \geqslant L_V\}] \\
&\qquad + \exp\{-\,\mathrm{const}\,|V|^\varepsilon\} \leqslant \mathrm{const}'\,|V|^{1+\varepsilon-\mu d}
\end{aligned}$$

by (1.2). Since the expression on the right is summable over $V(l)$ $(l \in V)$ for arbitrary $d > (1+\varepsilon)/\mu$, the Borel–Cantelli lemma implies (3.6) for $V(l)$ instead of $V$, as desired. □

## 4 The First $K$ Extreme Eigenvalues, $K = \mathrm{O}(|V|^\varepsilon)$

In this section, we study the asymptotic structure of the first $K = \mathrm{O}(|V|^\varepsilon)$ extreme eigenvalues $\lambda_{k,V}$ and localization properties of the corresponding (normalized) eigenfunctions $\psi(\cdot; \lambda_{k,V})$ of the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi(\cdot)$ in $l^2(V)$ when $V \uparrow \mathbb{Z}^\nu$ and $0 < \varepsilon < \frac{1}{2}$. In Theorem 4.1 we assume that the distribution function $F(\cdot)$ of $\xi(0)$ satisfies conditions (1.1) and (1.2). At the end of the section, we consider the case of (2.1) with sufficiently large $\rho/\kappa$ (Theorem 4.4). In both cases, a "typical" configuration $\{\xi^{(\omega)}(x) : x \in V\}$ possesses properties (2.5)–(2.8) for any large $V$ due to Lemmas 3.1 and 3.5. For such $\xi^{(\omega)}(\cdot)$, we apply finite-rank perturbation results (i.e., Theorem B.3) on the upper part of spectrum of $\mathcal{H}_V$.

To be more precise, let us introduce additional notation we use throughout Sect. 4. Fix $0 < \theta < \frac{1}{2}$ and write $L_{V,\theta} := f((1-\theta)\log|V|)$. Let the subset $\Pi_{V,\theta} \subset V$ be defined by (3.1), i.e., consisting of sites at which $\xi(\cdot)$ exceeds the level $L_{V,\theta}$. By $\widetilde{\xi}(\cdot)$ we denote the "noise" potential, viz. $\widetilde{\xi}(x) := \xi(x)$ if $\xi(x) < L_{V,\theta}$, and $\widetilde{\xi}(x) := 0$ otherwise. Abbreviate also

$$\xi^*(x) := \xi(x) \vee (L_{V,\theta} + 2\nu\kappa) \quad (x \in \mathbb{Z}^\nu).$$

Given $z \in V$, we now consider the principal eigenvalue $\widetilde{\lambda}(z)$ of the "single peak" Hamiltonian

$$\kappa\Delta_V + \widetilde{\xi}(\cdot) + \xi^*(z)\delta_z \quad \text{in } l^2(V);$$

here $\delta_z$ stands for the Kronecker symbol. As mentioned in Appendix B (Remark B.5), the eigenvalue $\widetilde{\lambda}(z)$ is the maximal solution of the dispersion equation

$$\widetilde{\mathcal{G}}_V(\lambda; z, z) = \frac{1}{\xi^*(z)}, \tag{4.1}$$

where $\widetilde{\mathcal{G}}_V(\lambda; \cdot, \cdot)$ is Green's function of the Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot)$ in $l^2(V)$. We expand $\widetilde{\mathcal{G}}_V(\lambda; z, z)$ over $\kappa\Delta$ (Lemma A.2 in Appendix A) to see that the solution $\widetilde{\lambda}(z)$ can be presented in series (2.10)–(2.12). The latter converges for each $\omega \in \Omega$, since $\xi^*(x) - \widetilde{\xi}(y) > 2\nu\kappa$ $(x \in V, y \in V)$ by the definition.

Now, let

$$\widetilde{\lambda}_{1,V} \geqslant \widetilde{\lambda}_{2,V} \geqslant \cdots \geqslant \widetilde{\lambda}_{|V|,V} \tag{4.2}$$

be the variational series of the sample $\{\widetilde{\lambda}(x) : x \in V\}$. Let $\{\tau(1), \tau(2), \ldots, \tau(|V|)\}$ be a (random) permutation of the numbers $\{1, 2, \ldots, |V|\}$ defined as follows. Write $\tau(0) := 0$ by convention, and for each $1 \leqslant K \leqslant |V|$,

$$\begin{aligned}\tau(K) := \tau_V(K) := \min\bigl\{1 \leqslant l \leqslant |V| : \widetilde{\lambda}(z_{l,V}) = \widetilde{\lambda}_{K,V} \text{ and} \\ l \neq \tau_V(k) \text{ for each } 0 \leqslant k \leqslant K-1\bigr\}.\end{aligned} \tag{4.3}$$

Here, remember, the sites $z_{l,V} \in V$ $(1 \leqslant l \leqslant |V|)$ are associated with the variational series based on the sample $\{\xi(x) : x \in V\}$; cf. (1.7)–(1.9). Thus

$$\widetilde{\lambda}(z_{\tau(K),V}) = \widetilde{\lambda}_{K,V} \quad \text{for all } 1 \leqslant K \leqslant |V|. \tag{4.4}$$

In order to stress the dependence of $\tau(K)$ on $V$, we frequently use notation $\tau_V(K)$ instead of $\tau(K)$. Write

$$J_V := [|V|^{(1+\theta)/\mu}] \quad \text{where } \mu > \frac{(1+\theta)\nu}{1-2\theta}. \tag{4.5}$$

**Theorem 4.1** *Fix $0 < \theta < \frac{1}{2}$. Assume that $F(\cdot)$ satisfies conditions (1.1)–(1.2) for some $\mu > (1+\theta)\nu/(1-2\theta)$. Then for any $\varepsilon \in (0, \theta)$ the following almost sure limits hold*:

$$\limsup_V \max_{1 \leqslant k \leqslant |V|^\varepsilon} \frac{\log|\lambda_{k,V} - \widetilde{\lambda}_{k,V}|}{J_V A_V(k)} \leqslant -2, \tag{4.6}$$

$$\liminf_V \min_{1 \leqslant k < l \leqslant |V|^\varepsilon} \frac{\log(\widetilde{\lambda}_{k,V} - \widetilde{\lambda}_{l,V})}{J_V} \geqslant -1$$

*and*

$$\limsup_V \max_{1 \leqslant k \leqslant |V|^\varepsilon} \max_{x \neq z_{\tau(k),V}} \frac{\log|\psi(x; \lambda_{k,V})|}{A_V(k)|x - z_{\tau(k),V}|} \leqslant -1,$$

*where, for all* $1 \leqslant k \leqslant |V|^{\varepsilon}$,

$$A_V(k) := \log\bigl(\widetilde{\lambda}_{k,V} - L_{V,\theta}\bigr) \geqslant \log(L_{V,\varepsilon'} - L_{V,\theta}) \to \infty$$

*as* $|V| \to \infty$, *for each* $\varepsilon' \in (\varepsilon, \theta)$.

**Corollary 4.2** *If the conditions of Theorem 4.1 are fulfilled and* $\varepsilon \in [0, \theta)$, *then the following almost sure assertions hold true.*

(i)

$$\lim_V \frac{\log \#\{\lambda \in \operatorname{Spect}(\mathcal{H}_V) \colon \lambda \geqslant L_{V,\varepsilon}\}}{\log |V|} = \varepsilon.$$

(ii) *For an arbitrary sequence* $\{K_V\} \subset \mathbb{N}$ *such that* $(\log K_V)/\log |V| \to \varepsilon$,

$$\lim_V \frac{\log \tau_V(K_V)}{\log |V|} = \varepsilon.$$

(iii) *For arbitrarily fixed* $K \in \mathbb{N}$ *if, in addition,* $f(t + C \log t) - f(t) \to 0$ *as* $t \to \infty$ *for some* $C > \frac{1}{K}$, *then*

$$\lim_V \bigl(\lambda_{K,V} - f(\log |V|)\bigr) = 0.$$

*Proof of Corollary 4.2* First, combining Lemma 3.1(i) with Theorem 4.1 we obtain assertion (i) which in turn yields assertion (ii). Finally, (iii) follows from (4.6), Lemma 3.1(i) and Remark 3.3. □

*Proof of Theorem 4.1* For arbitrarily fixed $\varepsilon' \in (\varepsilon, \theta)$, we define the subset $\widetilde{\Pi}_V \subset \Pi_{V,\theta}$ by

$$\widetilde{\Pi}_V := \{u \in \Pi_{V,\theta} \colon \widetilde{\lambda}(u) \geqslant L_{V,\varepsilon'} + 2\nu\kappa\}$$

for each $V \supset V_0$.

**Lemma 4.3** (cf. Lemma 3.5) *Let* $F(\cdot)$ *satisfy the conditions of Theorem 4.1. Then with probability* 1

$$\min\bigl\{|\widetilde{\lambda}(x) - \widetilde{\lambda}(y)| \colon x \in \widetilde{\Pi}_V, y \in \widetilde{\Pi}_V, x \neq y\bigr\} \geqslant \mathrm{e}^{-J_V/2} \tag{4.7}$$

*for any* $V \supset V_0$.

The lemma is shown below.

We now finish the proof of Theorem 4.1 by combining Lemmas 3.1(i) and 4.3 with Theorem B.3. By $\Omega' \in \mathcal{F}$ we denote the following event

$$\begin{aligned} \Omega' := \bigl\{\omega : &\text{ for any } \delta \in \{1/2, 1/3, 1/4, \ldots\} \text{ there is } V_0 = V_0(\omega; \delta, \varepsilon', \theta) \\ &\text{such that, for any } V \supset V_0, \text{ the sample } \{\xi^{(\omega)}(x) \colon x \in V\} \\ &\text{satisfies assumptions (B.25)–(B.29) of Theorem B.3} \\ &\text{with } L := L_{V,\theta},\ h := L_{V,\varepsilon'} - L_{V,\theta} \text{ and } K := \bigl[|V|^{\varepsilon}\bigr]\bigr\}. \end{aligned}$$

Noting that $L_{V,\varepsilon'} - L_{V,\theta} \to \infty$ as $|V| \to \infty$ and using Lemmas 3.1(i) and 4.3, we obtain that $\mathbb{P}(\Omega') = 1$. Consequently, Theorem B.3 immediately implies the assertions of Theorem 4.1. $\square$

*Proof of Lemma 4.3* For each $z \in V$, let $\lambda^{(J_V)}(z)$ be the principal eigenvalue of the Hamiltonian

$$\kappa\Delta_V + \sum_{y:\, 1\leqslant|y-z|\leqslant J_V} \widetilde{\xi}(y)\delta_y + \xi^*(z)\delta_z \quad \text{in } l^2(V).$$

We now turn to equation (4.1) for the eigenvalue $\widetilde{\lambda}(z)$, and expand the left-hand side of (4.1) over $\kappa\Delta$ as in (2.10)–(2.12) (and the same for the eigenvalue $\lambda^{(J_V)}(z)$); $z \in \widetilde{\Pi}_V$. From this, it is straightforward to obtain with probability one that

$$\big|\widetilde{\lambda}(x) - \lambda^{(J_V)}(x)\big| \leqslant \text{const}\bigg(\frac{2\nu\kappa}{\lambda^{(J_V)}(x) - L_{V,\theta}}\bigg)^{2J_V-1} \quad \text{for any } x \in \widetilde{\Pi}_V, \tag{4.8}$$

for any $V \supset V_0$.

To show (4.7), we fix $\widetilde{\varepsilon} \in (\varepsilon', \theta)$ and abbreviate

$$S_V := \min\big\{|\lambda^{(J_V)}(x) - \lambda^{(J_V)}(y)| : x \in \Pi_{V,\tilde{\varepsilon}},\ y \in \Pi_{V,\tilde{\varepsilon}},\ x \neq y\big\},$$

and $e_V := \exp\{-\frac{1}{4}J_V\}$. Because of (4.8) we note that (4.7) is a consequence of the following estimate a.s.

$$S_V \geqslant e_V \quad \text{for any } V \supset V_0. \tag{4.9}$$

Let us prove (4.9). According to Remark 3.2 we have that

$$\mathbb{P}(S_V < e_V) \leqslant {\sum}' \mathbb{P}(S_V < e_V, \Pi_{V,\tilde{\varepsilon}} = V') + |V|^{-\text{const}} \tag{4.10}$$

for some const $> 0$ and for each $V \supset V_0$, where the summation $\sum'$ is taken over all subsets $V' \subset V$ satisfying the following two conditions: first, $\frac{1}{2}|V|^{\tilde{\varepsilon}} \leqslant |V'| \leqslant 2|V|^{\tilde{\varepsilon}}$ and, second, the minimum distance between sites in $V'$ is larger than $|V|^{(1-\tilde{\varepsilon}-\theta)/\nu}$. Noting that $\lambda^{(J_V)}(x)$ is a function of the variables $\xi(u)$ ($|u-x| \leqslant J_V$) and recalling the definition of $J_V$ (4.5), we see that the random variables $\lambda^{(J_V)}(x)$ and $\lambda^{(J_V)}(y)$ are independent, provided $x \in V'$, $y \in V' \setminus \{x\}$. As in (3.9), we then get that

$$\begin{aligned}
&\mathbb{P}(S_V < e_V, \Pi_{V,\tilde{\varepsilon}} = V') \\
&\quad \leqslant \text{const}\,|V|^2 \mathbb{P}\big(|\lambda^0 - \lambda^z| < e_V, \xi(0) \geqslant L_{V,\tilde{\varepsilon}}, \xi(z) \geqslant L_{V,\tilde{\varepsilon}}\big)\mathbb{P}(\Pi_{V,\tilde{\varepsilon}} = V'),
\end{aligned} \tag{4.11}$$

with $\lambda^0 := \lambda^{(J_V)}(0)$ and $\lambda^z := \lambda^{(J_V)}(z)$ for $|z| > J_V$. Let $\mathcal{G}_V^{(z,J_V)}(\lambda; \cdot, \cdot)$ be Green's function of the Hamiltonian

$$\kappa\Delta_V + \sum_{y:\, 1\leqslant|y-z|\leqslant J_V} \widetilde{\xi}(y)\delta_y \quad \text{in } l^2(V).$$

Using the resolvent identity

$$\mathcal{G}_V^{(z,J_V)}(\lambda^z; z, z) - \mathcal{G}_V^{(z,J_V)}(\lambda^0; z, z) = (\lambda^0 - \lambda^z)\mathcal{G}_V^{(z,J_V)}(\lambda^0)\mathcal{G}_V^{(z,J_V)}(\lambda^z)\delta_z(z)$$

combined with the equation

$$\mathcal{G}_V^{(z,J_V)}(\lambda; z, z) = \frac{1}{\xi^*(z)}$$

for $\lambda^z$, we obtain that

$$\left|\xi^*(z) - 1/\mathcal{G}_V^{(z,J_V)}(\lambda^0; z, z)\right| \leqslant \mathrm{const}\,|\lambda^0 - \lambda^z|$$

for $V \supset V_0$. Substituting this into the right-hand side of (4.11) and combining the latter with (4.10), we find that

$$\begin{aligned}
&\mathbb{P}(S_V < e_V) \\
&\quad \leqslant \mathrm{const}\,|V|^2 \mathbb{E}\big[\big(F\big(1/\mathcal{G}_V^{(z,J_V)}(\lambda^0; z, z) + \mathrm{const}\, e_V\big) \\
&\qquad - F\big(1/\mathcal{G}_V^{(z,J_V)}(\lambda^0; z, z) - \mathrm{const}\, e_V\big)\big)\mathbb{1}\{\xi(0) \geqslant L_{V,\tilde{\varepsilon}}\}\big] + |V|^{-\mathrm{const}} \\
&\quad \leqslant \mathrm{const}'\,|V|^2 |\log e_V|^{-\mu} \mathbb{P}(\xi(0) \geqslant L_{V,\tilde{\varepsilon}}) + |V|^{-\mathrm{const}} \\
&\quad \leqslant \mathrm{const}''\,|V|^{\tilde{\varepsilon}-\theta} + |V|^{-\mathrm{const}};
\end{aligned} \tag{4.12}$$

here the second estimate follows from the log-Hölder continuity of $F(\cdot)$. Pick a subsequence of torus $\{V(l) \colon l \in \mathbb{N}\}$ satisfying (3.7). Since the expression on the right of (4.12) is summable over $\{V(l)\}$, the Borel–Cantelli lemma implies (4.9) for $V(l)$. This is easily extended to an arbitrary sequence $\{V\}$. Lemma 4.3 is proved. □

We end this section with a discussion on the case when (2.1) holds, i.e.,

$$\lim_{t\to\infty}\big(f(t) - f(\delta t)\big) = -\rho \log\delta \quad \text{for all } 0 < \delta < 1, \tag{4.13}$$

provided the ratio $\rho/\kappa$ is large enough. Recall that, for arbitrary $0 \leqslant \rho \leqslant \infty$, condition (4.13) ensures the a.s. second-order asymptotics for the principal eigenvalue $\lambda_{1,V}$ (Proposition 2.1 in Sect. 2.1). As in the proof of Theorem 4.1, we obtain the following theorem.

**Theorem 4.4** *Fix constants $\kappa > 0$, $\rho > 0$, $0 < \theta < \frac{1}{2}$ and $0 < \varepsilon < \theta$ such that the constant*

$$\underline{A}(\rho, \kappa, \theta, \varepsilon) := \log\left(\frac{1}{2\nu}\frac{\rho}{\kappa}\log\frac{1-\varepsilon}{1-\theta}\right) > 0 \quad \textit{is large enough}$$

*(say, $\underline{A}(\rho, \kappa, \theta, \varepsilon) \geqslant \log(36\nu)$). Assume that $F(\cdot)$ satisfies conditions (4.13) and (1.2) for some $\mu > (1+\theta)\nu/(1-2\theta)$. Then with probability one the assertions of Theorem 4.1 are fulfilled with*

$$A_V(k) := \log\frac{\widetilde{\lambda}_{k,V} - L_{V,\theta}}{2\nu\kappa} \geqslant \underline{A}(\rho, \kappa, \theta, \varepsilon) \quad \textit{for any } 1 \leqslant k \leqslant |V|^{\varepsilon}.$$

Recall that, under the conditions of Theorem 4.1 (i.e., the case of (4.13) with $\rho = \infty$), with probability one $\log \tau_V(K) = \mathrm{o}(\log|V|)$ when $|V| \to \infty$ and $K \in \mathbb{N}$ is fixed (see Corollary 4.2(ii) with $\varepsilon = 0$). Theorem 4.4 tells us that in the case of (4.13) with $\rho/\kappa$ large enough, the eigenvalue $\lambda_{K,V}$ still corresponds to an isolated $\xi_V$-peak. This statement is now specified more precisely in the following corollary.

**Corollary 4.5** *Under the conditions of Theorem 4.4 for some* $0 < \theta < \frac{1}{2}$, *we have the following limits in probability for fixed* $K \in \mathbb{N}$:

$$\lim_{\rho/\kappa\to\infty} \lim_{V} \big(\lambda_{K,V} - f(\log|V|)\big)\frac{1}{\rho}\left(\frac{\rho}{\kappa}\right)^2 \log\frac{\rho}{\kappa} = \nu \tag{4.14}$$

*and*

$$\lim_{\rho/\kappa\to\infty} \lim_{V} \frac{\log\tau_V(K)}{\log|V|}\left(\frac{\rho}{\kappa}\log\frac{\rho}{\kappa}\right)^2 = \frac{\nu}{2}.$$

*Sketch of the proof* We show that the first two asymptotic terms of $\lambda_{K,V}$ are carried by a $\xi_V$-peak and its nearest neighbor values.

Since our equation $\kappa\Delta\psi + \xi(\cdot)\psi = \lambda\psi$ is reduced to the new one: $\kappa'\Delta\psi + \xi'(\cdot)\psi = \lambda'\psi$, where $\kappa' := \kappa/\rho$, $\lambda' := \lambda/\rho$ and the distribution function of $\xi'(0)$ satisfies (4.13) with $\rho = 1$, it suffices therefore to show Corollary 4.5 for $\rho = 1$, i.e., when $\kappa \to 0$.

Using condition (4.13) and taking into account Remark 3.3 and the proof of (2.16), we obtain that a.s.

$$|\lambda_{K,V} - f(\log|V|)| < 3\nu\kappa \tag{4.15}$$

for fixed $K \in \mathbb{N}$, $\kappa > 0$ and each $V \supset V_0(\kappa)$. According to Theorem 4.4, we may expand $\lambda_{K,V}$ as in (2.9)–(2.12) and then apply (4.15) and (4.13) with $\rho = 1$ to estimate the terms of this expansion. We therefore obtain the following estimates a.s.

$$\zeta^{(-)}_{K,V} - \mathrm{const}\,\kappa^3 \leqslant \lambda_{K,V} \leqslant \zeta^{(+)}_{K,V} + \mathrm{const}\,\kappa^3 \tag{4.16}$$

for fixed $K \in \mathbb{N}$, small $\kappa$ and each $V \supset V_0(\kappa)$; here const $> 0$ does not depend on $\kappa$ and

$$\zeta^{(\pm)}(x) := \xi(x) + \frac{2\nu\kappa^2}{f(\log|V|) - \xi^{(\pm)}(x)\wedge L_{V,\theta}} \quad (x \in V),$$

where $\xi^{(+)}(x) := \max_{|y|=1}\xi(y+x)$ and $\xi^{(-)}(x) := \min_{|y|=1}\xi(y+x)$. By $f^{(\pm)}(\cdot)$ denote the inverse function of $-\log\mathbb{P}(\xi^{(\pm)}(0) > t)$, $t \in \mathbb{R}$. From (4.13) with $\rho = 1$ we have that

$$f^{(+)}(t) - f(t) \to 0 \quad \text{and} \quad f^{(-)}(t) - f(t) \to -\log(2\nu) \quad \text{as } t \to \infty. \tag{4.17}$$

In view of (4.16), it suffices to show assertion (4.14) ($\rho = 1$) for $\zeta^{(\pm)}_{K,V}$ instead of $\lambda_{K,V}$. Let us consider $\zeta^{(+)}_{1,V}$. Pick up a sequence of partitions $0 =: \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n := 1/3$ ($n \in \mathbb{N}$) of the interval $[0, 1/3]$ such that $\max_{i\leqslant n}(\varepsilon_i - \varepsilon_{i-1}) \to 0$ as $n \to \infty$. Clearly a.s.

$$\begin{aligned}\zeta^{(+)}_{1,V} &= \max_{1\leqslant i\leqslant n} \max_{|V|^{\varepsilon_{i-1}}\leqslant k\leqslant |V|^{\varepsilon_i}} \zeta^{(+)}(z_{k,V}) \\ &\leqslant \max_{1\leqslant i\leqslant n}\left(\xi_{[|V|^{\varepsilon_{i-1}}],V} + \frac{2\nu\kappa^2}{f(\log|V|) - \max_{k\leqslant|V|^{\varepsilon_i}}\xi^{(+)}(z_{k,V})}\right)\end{aligned} \tag{4.18}$$

for $V \supset V_0$. To estimate the right-side of (4.18), we need additional statements which follow from more general results given in [6].

**Lemma 4.6** [6] *Fix a sequence* $\{K_V\}$ *such that* $K_V \to \infty$ *and* $K_V = \mathrm{O}(|V|^{\varepsilon})$ *(as* $|V| \to \infty$*) for some* $0 < \varepsilon < 1/2$*. Then*

(i)

$$\sqrt{K_V} \max_{K_V \leqslant l \leqslant |V|} \left| \eta_{l,V} - \log \frac{|V|}{l} \right| = \mathrm{O}(1) \quad \textit{in probability}$$

*and*

(ii)

$$\max_{1 \leqslant l \leqslant K_V} \xi^{(\pm)}\left(z_{l,V}\right) = f^{(\pm)}(\log K_V + \mathrm{O}(1)) \quad \textit{in probability}$$

*as* $|V| \to \infty$*.*

Now, the first summand (resp., the second summand) under $\max_{i \leqslant n}$ in (4.18) is estimated by applying Lemma 4.6(i) with $K_V = |V|^{\varepsilon_{i-1}}$ (resp., Lemma 4.6(ii) with $K_V = |V|^{\varepsilon_i}$), where, as before, we use formulas (1.7)–(1.9). In view of (4.13) and the first limit in (4.17), we therefore obtain that the right-hand side of (4.18) is equal to

$$f(\log |V|) + \max_{i \leqslant n} \left( \log(1 - \varepsilon_{i-1}) - \frac{2\nu\kappa^2}{\log \varepsilon_i} \right) + \mathrm{o}(1) \quad \text{as } |V| \to \infty$$

in probability. Moreover, the last limit may be taken uniformly in partitions $\{\varepsilon_i : 0 \leq i \leq n\}$ because of the fact that $\exp\{f(\cdot)\}$ is a regularly varying function with exponent 1 and, consequently, $f(t) - f(\delta t) \to -\log \delta$ (as $t \to \infty$) uniformly on each close subinterval of (0, 1). Summarizing these assertions, we obtain that, as $|V| \to \infty$,

$$\zeta_{1,V}^{(+)} - f(\log |V|) \to \sup_{0 < \varepsilon \leqslant 1/3} g^{(+)}(\varepsilon)$$

in probability, where $g^{(+)}(\varepsilon) := \log(1 - \varepsilon) - 2\nu\kappa^2 / \log \varepsilon$ and $\kappa$ is small enough. Similarly we have that

$$\zeta_{K,V}^{(-)} - f(\log |V|) \to \sup_{0 < \varepsilon \leqslant 1/3} g^{(-)}(\varepsilon)$$

in probability, where $g^{(-)}(\varepsilon) := \log(1 - \varepsilon) - 2\nu\kappa^2 / \log(\frac{\varepsilon}{2\nu})$ and $\kappa$ is small enough. The functions $g^{(\pm)}(\cdot)$ attain their maximum at the points $\varepsilon^* = \varepsilon^*(\kappa) \in (0, 1/3)$ such that

$$\varepsilon^* = \frac{\nu}{2} \left( \frac{\kappa}{\log \kappa} \right)^2 (1 + \mathrm{o}(1)) \quad \text{and} \quad g^{(\pm)}(\varepsilon^*) = -\frac{\nu\kappa^2}{\log \kappa} (1 + \mathrm{o}(1)) \tag{4.19}$$

as $\kappa \downarrow 0$. Thus, (4.14) is proved.

These arguments imply also that the random functions $\zeta^{(\pm)}(z_{k,V})$ and thus $\widetilde{\lambda}(z_{k,V})$ $(1 \leqslant k \leqslant |V|^{1/3})$ attain their maximum at $k_V^*$ such that $\log k_V^* / \log |V| = \varepsilon^*(\kappa) + \mathrm{o}(1)$ as $|V| \to \infty$ in probability, for $\kappa$ small enough. This together with the first limit in (4.19) concludes the proof of the second assertion of Corollary 4.5. □

## 5 Extremal Type Limit Theorems for the First $K$ Eigenvalues, $K$ Fixed

In this section, we study joint limit distributions of a finite number of the first extreme eigenvalues $\lambda_{K,V}$. In Sect. 5.1, we consider the case when the distribution function $F(\cdot)$ satisfies conditions (1.1) and (1.2).

In Sect. 5.2, we briefly discuss the case of $-\log(1-F(t)) = \mathrm{o}(t^3)$, i.e., a potential with extremely sharp peaks. This condition implies that the main contribution to the extreme eigenvalue $\lambda_{K,V} = \lambda(z_{K,V})$ comes from the first two terms of expansion (2.9)–(2.12) with $z^0 = z_{K,V}$; i.e., $\lambda_{K,V} = \xi_{K,V} + 2\nu\kappa^2/\xi_{K,V} + \mathrm{O}(1/\xi^2_{K,V})$. Therefore, in extremal limit theorems, the normalizing constants for $\lambda_{K,V}$ may be chosen the same as those for $\xi_{K,V} + 2\nu\kappa^2/\xi_{K,V}$. This is illustrated by an example with polynomially decaying tails of distributions.

### 5.1 General Conditions on Distributions

The main result of this subsection is Theorem 5.2 concerning joint limit distributions for the normalized eigenvalues $(\lambda_{K,V} - B_V)A_V$ and their spacings $(\lambda_{K,V} - \lambda_{K+1,V})A_V$ for fixed $K \in \mathbb{N}$. The normalizing constants $A_V > 0$ and $B_V \in \mathbb{R}$ are expressed in terms of distributional tails of the random variable $\widetilde{\lambda}(z)$. Recall that, for each $z \in V$, $\widetilde{\lambda}(z)$ is the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot) + \xi^*(z)\delta_z$ in $l^2(V)$ and $\widetilde{\lambda}(z)$ is expanded in series (2.12) over $\widetilde{\xi}(x)/\xi^*(z)$, $x \in V$ (see Sect. 4).

Extremal type limit theorems for $\xi_{K,V}$ (i.e., the case $\kappa = 0$ in our model) are well known in mathematical literature (e.g., [19, 33, 40]). Let $\operatorname{esssup}\xi(0) = \infty$, and assume that there exist constants $a_V > 0$ and $b_V \in \mathbb{R}$ such that the sequence of distributions

$$\mathbb{P}\big((\xi_{1,V} - b_V)a_V \leqslant t\big) = F^{|V|}(b_V + t/a_V), \quad t \in \mathbb{R},$$

converges (as $|V| \to \infty$) to a nondegenerate distribution function $G(\cdot)$ at any continuity point of $G(\cdot)$. ($G(\cdot)$ is called degenerate if $G(\cdot)$ is an indicator function of interval). Then $G(\cdot)$ must be one of the following max-stable distributions:

$$G_\beta(t) = \begin{cases} \exp\{-t^{-\beta}\} & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0, \end{cases} \tag{5.1}$$

for some $\beta > 0$, and

$$G_{\exp}(t) = \exp\{-\mathrm{e}^{-t}\}, \quad t \in \mathbb{R} \tag{5.2}$$

(up to scale transformations $G_\beta(at+b)$ and $G_{\exp}(a't+b')$, $t \in \mathbb{R}$, for some constants $a > 0$, $a' > 0$, $b \in \mathbb{R}$ and $b' \in \mathbb{R}$). Moreover, in the case of limit $G_\beta(\cdot)$ we may choose $b_V \equiv 0$. See Chap. 1 of [33] for a detailed discussion on the subject.

Let us introduce the following functions

$$G_\beta(s_1, \ldots, s_{K-1}; s) = \frac{\beta}{(K-1)!}\left(\prod_{l=1}^{K-1} s_l^{-l\beta}\right)\int_s^\infty v^{-K\beta-1}\exp\{-v^{-\beta}\}\,\mathrm{d}v \tag{5.3}$$

with $s_l \geqslant 1$ $(1 \leqslant l \leqslant K-1)$, $s \geqslant 0$, and

$$G_{\exp}(t_1, \ldots, t_{K-1}; t) = \frac{1}{(K-1)!}\left(\prod_{l=1}^{K-1} \mathrm{e}^{-lt_l}\right)\int_t^\infty \exp\{-Kv - \mathrm{e}^{-v}\}\,\mathrm{d}v \tag{5.4}$$

with $t_l \geqslant 0$ $(1 \leqslant l \leqslant K-1)$ and $t \in \mathbb{R}$. Here $\prod_{l=1}^{0} \ldots := 1$.

**Theorem 5.1** (see [33]) (i) *If there are constants $a_V > 0$ such that*

$$\lim_V |V|\big(1 - F(s/a_V)\big) = -\log G_\beta(s)$$

*for each $s \geqslant 0$, then*

$$\lim_V \mathbb{P}\left(\frac{\xi_{1,V}}{\xi_{2,V}} > s_1, \ldots, \frac{\xi_{K-1,V}}{\xi_{K,V}} > s_{K-1}, \xi_{K,V} a_V > s\right) = G_\beta(s_1, \ldots, s_{K-1}; s)$$

*for any $s_l \geqslant 1$ $(1 \leqslant l \leqslant K-1)$ and any $s \geqslant 0$.*

(ii) *If there are constants $a_V > 0$ and $b_V \in \mathbb{R}$ such that*

$$\lim_V |V|\big(1 - F(b_V + t/a_V)\big) = -\log G_{\exp}(t)$$

*for each $t \in \mathbb{R}$, then*

$$\lim_V \mathbb{P}\big((\xi_{1,V} - \xi_{2,V})a_V > t_1, \ldots, (\xi_{K-1,V} - \xi_{K,V})a_V > t_{K-1}, (\xi_{K,V} - b_V)a_V > t\big) = G_{\exp}(t_1, \ldots, t_{K-1}; t)$$

*for any $t_l \geqslant 0$ $(1 \leqslant l \leqslant K-1)$ and $t \in \mathbb{R}$.*

We now turn to extremal type limit theorems for the eigenvalues $\lambda_{K,V}$. Let $\widetilde{\lambda}(z)$ be the principal eigenvalue of the "single peak" Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot) + \xi^*(z)\delta_z$; $z \in V$. From Lemma 7(ii) of [5] we know that if, for some constants $A_V > 0$ and $B_V \in \mathbb{R}$, the sequence of distributions $\mathbb{P}^{|V|}(\widetilde{\lambda}(0) \leqslant B_V + t/A_V)$, $t \in \mathbb{R}$, has (as $|V| \to \infty$) a nondegenerate limit $G(\cdot)$, then $G(\cdot)$ must be one of max-stable distributions (5.1) and (5.2) (up to scale transformation).

**Theorem 5.2** (cf. Theorem 4 of [5]) *Assume that $F(\cdot)$ satisfies conditions* (1.1) *and* (1.2) *for some constants $\mu > (1+\theta)\nu/(1-2\theta)$ and $0 < \theta < 1/2$.*

(i) *If there are constants $A_V > 0$ such that*

$$\lim_V |V|\mathbb{P}\big(\widetilde{\lambda}(0) > s/A_V\big) = -\log G_\beta(s)$$

*for each $s \geqslant 0$, then*

$$\lim_V \mathbb{P}\left(\frac{\lambda_{1,V}}{\lambda_{2,V}} > s_1, \ldots, \frac{\lambda_{K-1,V}}{\lambda_{K,V}} > s_{K-1}, \lambda_{K,V} A_V > s\right) = G_\beta(s_1, \ldots, s_{K-1}; s) \tag{5.5}$$

*for any $s_l \geqslant 1$ $(1 \leqslant l \leqslant K-1)$ and any $s \geqslant 0$.*

(ii) *If there are constants $A_V > 0$ and $B_V \in \mathbb{R}$ such that*

$$\lim_V |V|\mathbb{P}\big(\widetilde{\lambda}(0) > B_V + t/A_V\big) = -\log G_{\exp}(t) \tag{5.6}$$

*for each* $t \in \mathbb{R}$, *then*

$$\lim_V \mathbb{P}\big((\lambda_{1,V} - \lambda_{2,V})A_V > t_1, \dots, (\lambda_{K-1,V} - \lambda_{K,V})A_V > t_{K-1}, (\lambda_{K,V} - B_V)A_V > t\big) = G_{\exp}(t_1, \dots, t_{K-1}; t) \tag{5.7}$$

*for any* $t_l \geqslant 0$ $(1 \leqslant l \leqslant K-1)$ *and* $t \in \mathbb{R}$.

*Proof* It suffices to prove part (ii), since the proof of part (i) is similar. Using the notation of Sect. 4 (in particular, $J_V \in \mathbb{N}$ given by (4.5)), we again consider the principal eigenvalue $\lambda^{(J_V)}(z)$ of the Hamiltonian $\kappa\Delta_V + \sum_{y:\, 1\leqslant|y-z|\leqslant J_V} \widetilde{\xi}(y)\delta_y + \xi^*(z)\delta_z$ in $l^2(V)$; $z \in V$. Let $\lambda_{1,V}^{(J_V)} \geqslant \lambda_{2,V}^{(J_V)} \geqslant \cdots \geqslant \lambda_{|V|,V}^{(J_V)}$ be the variational series of the sample $\{\lambda^{(J_V)}(z)\colon z \in V\}$.

**Lemma 5.3** *Under the conditions of Theorem* 5.2(ii), *we have extremal type limit theorem* (5.7) *for* $\lambda_{l,V}^{(J_V)}$ *replacing* $\lambda_{l,V}$ $(1 \leqslant l \leqslant K)$.

Lemma 5.3 is proved below.

Theorem 4.1 and (4.8) imply that a.s.

$$\max_{1\leqslant k\leqslant |V|^{\varepsilon}} |\lambda_{k,V} - \lambda_{k,V}^{(J_V)}| = \mathrm{O}\left(\frac{\mathrm{const}}{L_{V,\varepsilon} - L_{V,\theta}}\right)^{2J_V-1} \tag{5.8}$$

as $|V| \to \infty$, for $0 < \varepsilon < \theta$. We now obtain the assertion of Theorem 5.2(ii) by combining Lemma 5.3, limit (5.8) and the following lemma:

**Lemma 5.4** *If the conditions of Theorem* 5.2(ii) *are fulfilled*, *then*

$$\limsup_V \frac{\log A_V}{J_V} \leqslant 0.$$

It remains to prove Lemmas 5.3 and 5.4.

*Proof of Lemma 5.3* It suffices to check that $\lambda^{(J_V)}(\cdot)$ satisfies Leadbetter's mixing conditions slightly modified for random fields. We first note that $\lambda^{(J_V)}(x)$ $(x \in \mathbb{Z}^\nu)$ form an array of identically distributed random variables with dependence range $\mathrm{const}\, J_V = \mathrm{o}(|V|^{1/\nu})$. Therefore, according to Theorem 5.7.2 in [33], we only need to check the following (local dependence) condition:

$$\limsup_V \sum_{y\in V} \sum_{0<|x|<(|V|/K)^{1/\nu}} \mathbb{P}\left(\lambda^{(J_V)}(y) > u_V, \lambda^{(J_V)}(y+x) > u_V\right) \to 0 \quad \text{as } K \to \infty, \tag{5.9}$$

with $u_V := B_V + A_V^{-1}t$ and fixed $t \in \mathbb{R}$. Taking into account the definition of the variables $\lambda^{(J_V)}(\cdot)$ and noting that $u_V - L_{V,\theta} \to \infty$ as $|V| \to \infty$, we obtain that the double sum under the limit in (5.9) does not exceed

$$|V| \sum_{0<|x|\leqslant J_V} \mathbb{P}\big(\lambda^{(J_V)}(0) > u_V, \lambda^{(J_V)}(x) > u_V\big) + \frac{\mathrm{const}}{K}\big(|V|\mathbb{P}(\lambda^{(J_V)}(0) > u_V)\big)^2$$

$$\leqslant |V| \sum_{0<|x|\leqslant J_V} \mathbb{P}\big(\xi(0) > L_{V,\theta}, \xi(x) > L_{V,\theta}\big) + \frac{\mathrm{const}'}{K}$$
$$\leqslant \mathrm{const}''\, J_V^{\nu} |V|^{2\theta-1} + \mathrm{const}'\, K^{-1},$$

where $J_V^{\nu} = \mathrm{o}(|V|^{1-2\theta})$ by (4.5). Thus, passing to the limits, as first $|V| \to \infty$ and then $K \to \infty$, we arrive at (5.9), as claimed. □

*Proof of Lemma 5.4* Fix an arbitrary $t > 0$, and write $T_V := t/A_V$. Assume for a moment that $T_{V'} < \exp\{-\mathrm{const}\, J_{V'}\}$ for some subsequence $\{V'\}$, $|V'| \to \infty$. Then, arguing as in the proof of Lemma 4.3,

$$\mathbb{P}\big(|\lambda_{1,V'}^{(J_{V'})} - B_{V'}| < T_{V'}\big) \leqslant |V'|^{-\mathrm{const}} \to 0$$

as $|V'| \to \infty$. This contradicts to the nondegenerity of $G(\cdot)$, and, therefore, Lemma 5.4 is proved. □

## 5.2 Distributions with Heavy Tails (Discussion)

In this section, we analyze the first few asymptotical terms of $\lambda_{K,V} = \lambda(z_{K,V})$ in expansion (2.9)–(2.12) with $z^0 = z_{K,V}$ for fixed $K \in \mathbb{N}$.

For $p \geqslant 0$ and $c \geqslant 0$, we introduce the function $f(s; p; c) := f(s)^p (f(s+c) - f(s))$, $s > 0$. The condition

$$\lim_{s\to\infty} f(s; p; c) = \infty \quad \text{for any } 0 < c < 1 \tag{5.10}$$

implies that

$$\lim_V \xi_{K+1,V}^p (\xi_{K,V} - \xi_{K+1,V}) = \infty \quad \text{in probability} \tag{5.11}$$

for fixed $K \in \mathbb{N}$. We impose the additional condition (stronger than (1.1))

$$\limsup_{s\to\infty} \frac{f((1-\varepsilon)s)}{f(s)} < 1 \quad \text{for some } 0 < \varepsilon < \frac{1}{2} \tag{5.12}$$

to guarantee the distinct difference in height between the "noise" potential $\widetilde{\xi}(\cdot)$ and a peak $\xi_{K,V}$, viz.

$$\limsup_V \max_{x\in V} \widetilde{\xi}(x)/\xi_{K,V} < 1 \quad \text{in probability.} \tag{5.13}$$

Taking into account such a strongly pronounced asymptotic structure of $\xi_V$-peaks (i.e., limits (5.11), (5.13) and Lemma 3.1) and using the same arguments as in the proof of Theorems 4.1 and 5.2, we obtain the following statements illustrating very simple asymptotic structure of the extreme eigenvalues $\lambda_{K,V}$.

**Theorem 5.5** *Fix $K \in \mathbb{N}$. Assume that $f(\cdot)$ satisfies (5.12) and the following condition*

$$\lim_{s\to\infty} f(s; 1; c) = \infty \quad \textit{for any } 0 < c < 1.$$

*Then*

$$\limsup_{V} \xi_{K,V} |\lambda_{K,V} - \xi_{K,V}| < \text{const} \quad \textit{in probability} \tag{5.14}$$

*for some* (*nonrandom*) const $> 0$. *Moreover, by* (5.14) *and* (5.11) *with* $p = 1$ *we obtain limit theorem* (5.5) *for* $\lambda_{k,V}$ $(1 \leqslant k \leqslant K)$ *provided* $\lim_V |V|\mathbb{P}(\xi(0) > s/A_V) = -\log G_\beta(s)$ $(s \geqslant 0)$, *or limit theorem* (5.7) *for* $\lambda_{k,V}$ $(1 \leqslant k \leqslant K)$ *provided* $\lim_V |V|\mathbb{P}(\xi(0) > B_V + t/A_V) = -\log G_{\exp}(t)$ $(t \in \mathbb{R})$.

Write

$$\xi^{(0)}(x) := \xi(x) + \frac{2\nu\kappa^2}{\xi(x) \vee 1} \quad (x \in \mathbb{Z}^\nu).$$

**Theorem 5.6** *Fix* $K \in \mathbb{N}$. *Assume that* $f(\cdot)$ *satisfies* (5.12) *and the following condition*

$$\lim_{s\to\infty} \inf_{a\in(c,\theta s)} \left( f(s;2;a) - \frac{f(2a)}{c} \right) = \infty \quad \textit{for any } 0 < c < 1 \textit{ and some } \theta \in (\varepsilon, 1/2).$$

*Then*

$$\limsup_{V} \xi^2_{K,V} |\lambda_{K,V} - \xi^{(0)}_{K,V}| < \infty \quad \textit{in probability.} \tag{5.15}$$

*Moreover, by* (5.15) *and* (5.11) *with* $p = 2$ *we obtain limit theorem* (5.5) *for* $\lambda_{k,V}$ $(1 \leqslant k \leqslant K)$ *provided* $\lim_V |V|\mathbb{P}(\xi^{(0)}(0) > s/A_V) = -\log G_\beta(s)$ $(s \geqslant 0)$, *or limit theorem* (5.7) *for* $\lambda_{k,V}$ $(1 \leqslant k \leqslant K)$ *provided* $\lim_V |V|\mathbb{P}(\xi^{(0)}(0) > B_V + t/A_V) = -\log G_{\exp}(t)$ $(t \in \mathbb{R})$.

A detailed proof of Theorems 5.5, 5.6 and related statements will be carried out in our forthcoming papers [6] and [7].

*Remark 5.7* We note that, under the conditions of Theorem 5.5, the normalizing constants $A_V$ and $B_V$ for $\lambda_{K,V}$ are chosen the same as those for $\xi_{K,V}$.

*Remark 5.8* According to [6], for fixed $p \geqslant 0$, condition (5.10) implies that $-\log(1 - F(t)) = \mathrm{o}(t^{p+1})$ (as $t \to \infty$), i.e., heavy tails of the distribution function $F(\cdot)$. Note that, under the assumptions of Theorems 5.5 or 5.6, condition on log-Hölder continuity of $F(\cdot)$ is removed.

*Example 5.9* Assume that

$$1 - F(t) = t^{-\beta}(1 + \mathrm{o}(1)) \quad \text{as } t \to \infty,$$

where $\beta > 0$. In this case, $f(s) = \mathrm{e}^{s/\beta + \mathrm{o}(1)}$ (as $s \to \infty$) and, therefore, the conditions of Theorem 5.5 are fulfilled. Moreover, since $|V|(1 - F(|V|^{1/\beta} t)) \to t^{-\beta}$ for any $t > 0$, we obtain extremal type limit theorem (5.5) for $\lambda_{k,V}$ $(1 \leqslant k \leqslant K)$ with the normalizing constants $A_V = |V|^{-\beta}$. Note that $A_V$ coincide with the normalizing constants $a_V$ in the corresponding limit theorem for $\xi_{k,V}$ $(1 \leqslant k \leqslant K)$.

In Sect. 6, we consider the case of $F(\cdot)$ with fractional-exponential tails satisfying the conditions of Theorems 5.1(ii), 5.2(ii) and 5.6.

## 6 Extremal Type Limit Theorems: Distributions with Fractional-Exponential Tails

In this section, we assume that the (common) distribution function $F(\cdot)$ of the i.i.d. field $\xi(\cdot)$ satisfies the condition

$$1 - F(t) = t^{-\beta}\,\mathrm{e}^{-At^{\alpha}}(1 + \mathrm{o}(1)) \quad \text{as } t \to \infty, \tag{6.1}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ and $A > 0$. In the case of $\alpha \geqslant 3$, we additionally claim that $F(\cdot)$ has a density $p(t) := \frac{\mathrm{d}F(t)}{\mathrm{d}t}$ ($t \geqslant t_0$) such that

$$p(t) = A\alpha t^{\alpha-\beta-1}\,\mathrm{e}^{-At^{\alpha}}(1 + \mathrm{o}(1)) \quad \text{as } \ t \to \infty. \tag{6.2}$$

By the partial integration, condition (6.2) implies (6.1) for each $\alpha > 0$. Condition (6.1) in turn implies that

$$f(s) = (s/A)^{1/\alpha} - \frac{\beta}{\alpha^2 A}\frac{\log(s/A)}{(s/A)^{1-1/\alpha}} + \mathrm{o}\left(s^{-1+1/\alpha}\right) \quad \text{as } s \to \infty \tag{6.3}$$

for each $\alpha > 0$. Note that the case $\alpha = 2$, $\beta = 1$ and $A = (4\pi)^{-1}$ in (6.2) includes the distribution density of the Gaussian variable $\xi(0)$ with $\mathbb{E}\xi(0) = 0$ and $\mathbb{E}\xi(0)^2 = 2\pi$.

We now abbreviate

$$l_V = \left(\frac{\log|V|}{A}\right)^{1/\alpha}$$

and $J := [\alpha/2]$. For each $z \in V$, let $\lambda^{(J)}(z)$ be the principal eigenvalue of the Hamiltonian

$$\kappa\Delta_V + \sum_{y:\,1\leqslant|y-z|\leqslant J} \widetilde{\xi}(y)\delta_y + \xi^*(z)\delta_z \quad \text{in } l^2(V),$$

where $\widetilde{\xi}(x) := \xi(x)$ if $\xi(x) < (2/3)^{1/\alpha} l_V$ and $\widetilde{\xi}(x) := 0$ otherwise, and $\xi^*(\cdot) := \xi(\cdot) \vee ((3/4)^{1/\alpha} l_V)$ (cf. Sect. 4). For $\alpha \geqslant 3$, the distribution function $F(\cdot)$ satisfies the conditions of Theorem 4.1 for each $0 < \theta < 1/2$; therefore, for fixed $K \in \mathbb{N}$ we have with probability one that

$$\lambda_{K,V} = \lambda^{(J)}_{K,V} + \mathrm{O}(l_V^{-2J-1}) = l_V + \mathrm{O}(l_V^{-1}) \tag{6.4}$$

as $|V| \to \infty$, where the second limit follows from (1.7), (6.3) and Remark 3.3. For $\alpha < 3$, the function $f(\cdot)$ (6.3) satisfies the conditions of Theorem 5.6; therefore, for fixed $K \in \mathbb{N}$ we have the following limits in probability:

$$\lambda_{K,V} = \xi_{K,V} + \frac{2\nu\kappa^2}{\xi_{K,V}} + \mathrm{O}\left(l_V^{-2}\right) = l_V + \mathrm{O}\left(l_V^{-1}\right) + \mathrm{O}\left(\frac{\log l_V}{l_V^{\alpha-1}}\right) \tag{6.5}$$

as $|V| \to \infty$.

The purpose of the present section is to derive the asymptotic equations for the normalizing constants $A_V > 0$ and $B_V \in \mathbb{R}$ appearing in the limit (5.6), i.e., $\lim_V |V|\mathbb{P}(\lambda^{(J)}(0) > B_V + t/A_V) = \mathrm{e}^{-t}$ for each $t \in \mathbb{R}$. By Theorem 5.2(ii), this immediately implies limit theorem (5.7) for the (normalized) extreme eigenvalues $(\lambda_{K,V} - B_V)A_V$ and their spacings $(\lambda_{K,V} - \lambda_{K+1,V})A_V$. See Theorem 6.3 below, i.e., the main result of the present section.

To simplify the proceedings in the case of $\alpha \geqslant 3$, we need some additional notation and remarks. Write $j := [\alpha/2]$ if $\alpha - 2[\alpha/2] > 1$ and $j := [\alpha/2] - 1$ otherwise. Let $U :=$

$\{x : 1 \leqslant |x| \leqslant j\} \subset V$. We write $t_U := \{t_x : x \in U\} \in \mathbb{R}^{|U|}$. For $g : \mathbb{R}^{|U|} \to \mathbb{R}$, denote $g^{(x)} = \frac{\partial g}{\partial t_x}$, $g^{(xy)} = \frac{\partial^2 g}{\partial t_x \partial t_y}$, etc.

Let $\mathcal{G}_V^{(0,j)}(\widetilde{\xi}_U; \lambda; x, y) =: \mathcal{G}_V^{(0,j)}(\lambda; x, y)$ $(x \in V, y \in V)$ be Green's function of the Hamiltonian $\kappa\Delta_V + \widetilde{\xi}(\cdot)\mathbb{1}_U(\cdot)$ in $l^2(V)$. We then introduce the following functions:

$$g(t_U; \lambda) := \mathcal{G}_V^{(0,j)}(\widetilde{t}_U; \lambda; 0, 0) \tag{6.6}$$

and

$$Q(t_U; \lambda; \mu) := \left(\frac{\mu}{\lambda g(t_U; \lambda)}\right)^{\alpha} + \sum_{x \in U} (t_x)_+^{\alpha}, \tag{6.7}$$

for all $\lambda \geqslant l_V$, $\mu \geqslant l_V$ and all $t_U \in \mathbb{R}^{|U|}$; here $\widetilde{t}_x := t_x \mathbb{1}\{t_x < (2/3)^{1/\alpha} l_V\}$ $(x \in U)$ and $t_+ := t \vee 0$.

*Remark 6.1* In what follows, it is useful to expand $g(t_U; D_V)$ (6.6) over $\kappa\Delta$ as in Lemma A.2 with $D_V = l_V(1 + \mathrm{o}(1))$ instead of $\lambda$ and with $\widetilde{t}_x \mathbb{1}_U(x)$ instead of $\xi(x)$ for $x \in V$. Namely,

$$g(t_U; D_V) = \sum_{\Gamma} \kappa^{|\Gamma|} \prod_{v \in V} (D_V - \widetilde{t}_v \mathbb{1}_U(v))^{-n_v(\Gamma)}; \tag{6.8}$$

here the summation is taken over the nearest neighbor paths $\Gamma := \Gamma(0, 0)$ in $V$ (starting at 0 and ending at 0) of the length $|\Gamma| = \Sigma_v n_v(\Gamma) - 1 \geqslant 0$, where $n_v(\Gamma)$ denotes the number of times the path $\Gamma$ visits the site $v \in V$. The series in (6.8) converges, since the total number of paths $\Gamma$ with $|\Gamma| = k$ does not exceed $(2\nu)^{k-1}$. Clearly

$$g^{(x)}(t_U; D_V) = (D_V - \widetilde{t}_x)^{-1} \sum_{\Gamma = \Gamma_x} \kappa^{|\Gamma|} n_x(\Gamma) \prod_{v \in V} (D_V - \widetilde{t}_v \mathbb{1}_U(v))^{-n_v(\Gamma)} \tag{6.9}$$

$(x \in U)$; here the summation is taken over paths $\Gamma_x$ with $|\Gamma_x| \geqslant 2|x|$ visiting the site $x$. Therefore, as $|V| \to \infty$, the main asymptotic contribution $(\asymp l_V^{-2|x|-2})$ into (6.9) comes from the summands corresponding to $\Gamma_{x,\min}$ with $|\Gamma_{x,\min}| = 2|x|$, i.e., the shortest paths $\Gamma_x$ visiting $x$. By $c_x$ we denote the number of $\Gamma_{x,\min}$. Note that

$$g^{(x)}(t_U; D_V) = \kappa^{2|x|} c_x l_V^{-2|x|-2}(1 + \mathrm{o}(1)),$$

provided $0 \leqslant t_y = \mathrm{o}(l_V)$ for each $y \in U$.

For $\alpha \geqslant 3$ and $A > 0$, we write

$$\begin{aligned} C(\alpha) :=& \frac{|U|}{2} \log \frac{\alpha - 1}{2\pi A\alpha} \\ &- \begin{cases} \displaystyle\sum_{|x|=J} \log \mathbb{E} \exp\{A\alpha\kappa^{\alpha-1} c_x \xi(0)\} & \text{if } \alpha - 2[\frac{\alpha}{2}] = 1, \\ 0 & \text{otherwise;} \end{cases} \end{aligned}$$

here $c_x$ is specified in Remark 6.1.

Let us introduce the following system of equations:

$$AQ(T_U; B; B) - \log|V| + \beta \log l_V + \left(\beta - \frac{\alpha}{2}\right) \sum_{x \in U} \log T_x + C(\alpha) = 0 \tag{6.10}$$

and

$$Q^{(x)}(T_U; B; B) = 0 \quad \text{for any } x \in U. \tag{6.11}$$

If $\alpha = 3$, then $U = \varnothing$ and, therefore, the system of equations (6.10) and (6.11) reduces simply to equation (6.10) with $T_\varnothing := 0$ and $\Sigma_{x\in\varnothing} \ldots := 0$. For $\alpha > 3$ and for arbitrary $V \supset V_0$, the system of equations (6.10) and (6.11) has a solution $B_V \in \mathbb{R}_+$ and $T_{U;V} := \{T_{x;V} : x \in U\} \in \mathbb{R}_+^{|U|}$ such that

$$B_V = l_V + \mathrm{O}(l_V^{-1}) \tag{6.12}$$

and

$$T_{x;V} = \left(\kappa^{2|x|} c_x\right)^{\frac{1}{\alpha-1}} l_V^{1-\frac{2|x|}{\alpha-1}} (1 + \mathrm{o}(1)) \quad \text{for any } x \in U, \tag{6.13}$$

as $|V| \to \infty$ (to check this, use the assertions of Remark 6.1).

We finally abbreviate

$$a_V := \alpha A l_V^{\alpha-1} \quad \text{and} \quad b_V := l_V - (\beta \log l_V)/a_V, \tag{6.14}$$

and note that $\lim_V |V|(1 - F(b_V + a_V^{-1} t)) = \mathrm{e}^{-t}$ for each $t \in \mathbb{R}$. Therefore, by Theorem 5.1(ii) we obtain the following statement.

**Theorem 6.2** ($\kappa = 0$) *If $F(\cdot)$ satisfies condition (6.1) with $\alpha > 0$, then*

$$\begin{aligned}&\lim_V \mathbb{P}\big((\xi_{1,V} - \xi_{2,V})a_V > t_1, \ldots, (\xi_{K-1,V} - \xi_{K,V})a_V > t_{K-1}, (\xi_{K,V} - b_V)a_V > t\big)\\ &\quad = G_{\exp}(t_1, \ldots, t_{K-1}; t)\end{aligned}$$

*for each $t_l \geqslant 0$ $(1 \leqslant l \leqslant K-1)$ and each $t \in \mathbb{R}$, where $G_{\exp}(t_1, \ldots, t_{K-1}; t)$ is given by (5.4).*

With the previous notation and remarks, we are in a position to formulate the main result of the section.

**Theorem 6.3** *For $\alpha < 3$, let $F(\cdot)$ satisfy condition (6.1). For $\alpha \geqslant 3$, assume that $F(\cdot)$ has a density $p(t) := \frac{\mathrm{d}F(t)}{\mathrm{d}t}$ $(t \geqslant t_0)$ satisfying condition (6.2). Let constants $a_V$ and $b_V$ be defined by (6.14), and write:*

1) $B_V = b_V$ *if* $\alpha < 2$,
2) $B_V = b_V + 2\nu\kappa^2 l_V^{-1}$ *if* $2 \leqslant \alpha < 3$,
   *and*
3) *for* $\alpha \geqslant 3$, *constants* $B_V$ *are defined by the system of equations (6.10) and (6.11).*

*Then*

$$\begin{aligned}&\lim_V \mathbb{P}\big((\lambda_{1,V} - \lambda_{2,V})a_V > t_1, \ldots, (\lambda_{K-1,V} - \lambda_{K,V})a_V > t_{K-1}, (\lambda_{K,V} - B_V)a_V > t\big)\\ &\quad = G_{\exp}(t_1, \ldots, t_{K-1}; t)\end{aligned}$$

*for each $t_l \geqslant 0$ $(1 \leqslant l \leqslant K-1)$ and each $t \in \mathbb{R}$, where $G_{\exp}(t_1, \ldots, t_{K-1}; t)$ is given by (5.4).*

From (6.10) and (6.11) we now get the explicit expressions for the normalizing constants $B_V$ when $3 \leqslant \alpha < 4$ (cf. [9]).

**Corollary 6.4** *Under the conditions of Theorem 6.3 with $\alpha \geqslant 3$, the constants $B_V$ may be taken to be*:

(a) $B_V = b_V + c^{(0)} l_V^{-1} + c^{(3)} l_V^{1-\alpha}$ *if* $\alpha = 3$;

(b) $B_V = b_V + c^{(0)} l_V^{-1} + c_1^{(1)} l_V^{-\frac{\alpha+1}{\alpha-1}} + (c^{(2)} \log l_V + c^{(3)}) l_V^{1-\alpha}$ *if* $3 < \alpha < (3+\sqrt{17})/2$;

(c) $B_V = b_V + c^{(0)} l_V^{-1} + \sum_{i=1}^{2} c_i^{(1)} l_V^{-\frac{\alpha+2i-1}{\alpha-1}} + (c^{(2)} \log l_V + c^{(3)}) l_V^{1-\alpha}$ *if* $(3+\sqrt{17})/2 \leqslant \alpha < 4$;

*and moreover, as* $|V| \to \infty$,

(d) $B_V = l_V + c^{(0)} l_V^{-1} + \sum_{i=1}^{2} c_i^{(1)} l_V^{-\frac{\alpha+2i-1}{\alpha-1}} + \mathrm{O}(l_V^{-\frac{\alpha+5}{\alpha-1}}) + \mathrm{O}(l_V^{-3} \log l_V)$ *if* $\alpha \geqslant 4$.

*Here constants $c^{(i)}$ and $c_i^{(n)}$ are specified as follows*:

$$c^{(0)} := 2\nu\kappa^2, \qquad c_1^{(1)} := \frac{\alpha-1}{\alpha} 2\nu\kappa^{\frac{2\alpha}{\alpha-1}}, \qquad c_2^{(1)} := 2\nu\kappa^{\frac{2(\alpha+1)}{\alpha-1}},$$

$$c^{(2)} := \frac{\nu(\alpha-2\beta)(\alpha-3)}{A\alpha(\alpha-1)}$$

*and, finally, $c^{(3)}$ is defined by*

$$c^{(3)} := \begin{cases} \dfrac{2\nu}{A\alpha} \log \mathbb{E} \exp\{A\alpha\kappa^{\alpha-1}\xi(0)\} & \text{if } \alpha = 3, \\ \dfrac{\nu}{A\alpha}\left(\dfrac{2(\alpha-2\beta)\log\kappa}{\alpha-1} + \log\dfrac{2\pi A\alpha}{\alpha-1}\right) & \text{if } 3 < \alpha < 4. \end{cases}$$

Let us show Corollary 6.4. For $\alpha = 3$, by expanding $B_V g(0; B_V)$ in powers of $B_V$ and noting that $j = 0$, we rewrite equation (6.10) in the form:

$$A\left(\frac{B_V}{1 + c^{(0)} B_V^{-2}}\right)^3 - \log|V| + \beta \log l_V + C(3) = \varepsilon_V$$

for some $\varepsilon_V \to 0$. Iterating this with respect to $B_V$, we obtain expression (a) for $B_V$ with $\mathrm{o}(a_V^{-1})$ accuracy.

Let $3 < \alpha < 4$. In this case, we have that $j = 1$, $U = \{|x| = 1\}$ and, by symmetry, $T_{x;V} = T_{y;V}$ for all $x, y \in U$. As an initial approximation for $B_V$, we take $l_V$. Note that, by (6.12), $B_V - l_V = \mathrm{O}(l_V^{-1})$. Looking at $Q(T_U; B; B)$ (6.7) and expanding the expression

$$\big(B_V g(T_{U;V}; B_V)\big)^{-\alpha} = \big((l_V + B_V - l_V) g(T_{U;V}; l_V + B_V - l_V)\big)^{-\alpha}$$

in powers of $B_V - l_V$, we may therefore rewrite (6.10) and (6.11) in the form:

$$\begin{aligned} &A Q(T_{U;V}; l_V; B_V) - \log|V| + \beta \log l_V \\ &\quad + \left(\beta - \frac{\alpha}{2}\right) \sum_{|x|=1} \log T_{x;V} + C(\alpha) = \mathrm{O}(l_V^{\alpha-4}) \end{aligned} \tag{6.15}$$

and

$$Q^{(x)}(T_{U;V}; l_V; B_V) = \mathrm{O}(l_V^{\alpha-5}). \tag{6.16}$$

Rewrite (6.15) in the form

$$B_V^\alpha = (l_V g(T_{U;V}; l_V))^\alpha \biggl[ l_V^\alpha - \sum_{|x|=1} T_{x;V}^\alpha - \frac{\beta}{A}\log l_V - \frac{1}{A}\Bigl(\beta - \frac{\alpha}{2}\Bigr)\sum_{|x|=1}\log T_{x;V} - \frac{C(\alpha)}{A} + \mathrm{O}(l_V^{\alpha-4})\biggr]. \qquad (6.17)$$

We substitute this expression into (6.16) to eliminate $B_V$. From this, by using (6.13), we obtain the equation for $T_{x;V}$:

$$T_{x;V}^{\alpha-1} g(T_{U;V}; l_V)/g^{(x)}(T_{U;V}; l_V) = l_V^\alpha + \mathrm{O}(l_V^{\alpha-2}) \quad (|x|=1).$$

An iteration of the last identity with respect to $T_{x;V} = T_{x;V}(l_V)$ shows that

$$T_{x;V} = \sum_{n=1}^{2} e_n l_V^{\frac{\alpha-2n-1}{\alpha-1}} + \mathrm{O}\bigl(l_V^{\frac{\alpha-7}{\alpha-1}}\bigr) \quad (|x|=1);$$

here $e_1 := \kappa^{2/(\alpha-1)}$ and $e_2 := \kappa^{4/(\alpha-1)} 2(\alpha-1)^{-1}$. Substituting this in (6.17) and iterating the latter with respect to $B_V$, we obtain expressions (b) and (c) for $B_V$ with $\mathrm{o}(a_V^{-1})$ accuracy. The same is applied to derive (d). □

*Remark 6.5* (Localization properties of extreme eigenvalues $\lambda_{K,V}$) Assume that the conditions of Theorem 6.3 are fulfilled and $\alpha \geqslant 3$. As in Sect. 4, for arbitrarily fixed $K \in \mathbb{N}$, we define the random variables $\tau(K) := \tau_V(K) \in \{1, 2, \ldots, |V|\}$ by the equation $\lambda^{(J)}(z_{\tau(K),V}) = \lambda_{K,V}^{(J)}$, i.e., $z_{\tau(K),V} \in V$ is a location of the $K$th larger value $\lambda_{K,V}^{(J)}$ of the sample $\lambda^{(J)}(\cdot)$ in $V$ (note that the inequalities $\lambda_{1,V}^{(J)} > \lambda_{2,V}^{(J)} > \cdots > \lambda_{K,V}^{(J)}$ are strict with probability one). By combining (6.4) with Theorems 4.1 and 6.3, we obtain the following limits in probability [8]:

1) if $\alpha = 3$ then $\limsup_V \tau_V(K) < \infty$;
2) if $\alpha > 3$ then $\lim_V \frac{\log \tau_V(K)}{l_V^{\alpha(\alpha-3)/(\alpha-1)}} = 2\nu A\kappa^{2\alpha/(\alpha-1)}$.

*Proof of Theorem 6.3* For $\alpha < 3$, the assertion of the theorem follows from (6.5) and Theorem 6.2.

Let $\alpha > 3$. By the same arguments as in the proof of Lemma 5.3 combined with (6.4), it suffices to show that

$$\lim_V \mathbb{P}(\lambda_{1,V}^{(J)} \leqslant B_V + a_V^{-1} t) = \exp\{-\mathrm{e}^{-t}\} \quad \text{for all } t \in \mathbb{R}, \qquad (6.18)$$

where $\lambda_{1,V}^{(J)}$ denotes the maximum of the sample $\{\lambda^{(J)}(z) \colon z \in V\}$. Until the proof of Lemma 6.7, we abbreviate

$$\lambda^* := \lambda_{1,V}^{(J)} \quad \text{and} \quad \lambda^{(J)}(z^*) := \lambda^*,$$

i.e., $z^* \in V$ is a location of the eigenvalue $\lambda^*$. Recall that $\lambda^*$ is the maximal solution of the equation $\mathcal{G}_V^{(z^*,J)}(\lambda; z^*, z^*) = 1/\xi^*(z^*)$, where $\xi^*(\cdot)$ is given at the beginning of this section and $\mathcal{G}_V^{(z,J)}(\lambda; \cdot, \cdot)$ is Green's function of the Hamiltonian

$$\kappa\Delta_V + \sum_{y\colon 1\leqslant|y-z|\leqslant J} \widetilde{\xi}(y)\delta_y \quad \text{in } l^2(V).$$

The following two lemmas summarize the key steps in the proof of (6.18).

**Lemma 6.6** *We have with probability* 1 *that*

$$\max_{x\in V} \xi^*(x)\mathcal{G}_V^{(x,J)}(\lambda^*; x, x) = 1 \tag{6.19}$$

*for all* $V \supset V_0$.

*Proof* Clearly $\mathcal{G}_V^{(x,J)}(\mu; x, x)$ is nonincreasing in $\mu$ for each $x \in V \supset V_0$. Consequently

$$1 \equiv \xi^*(x)\mathcal{G}_V^{(x,J)}\big(\lambda^{(J)}(x); x, x\big) \geqslant \xi^*(x)\mathcal{G}_V^{(x,J)}(\lambda^*; x, x)$$

for each $x \in V \supset V_0$. On the other hand,

$$1 = \xi^*(z^*)\mathcal{G}_V^{(z^*,J)}(\lambda^*; z^*, z^*) \leqslant \max_{x\in V} \xi^*(x)\mathcal{G}_V^{(x,J)}(\lambda^*; x, x),$$

as claimed. Lemma 6.6 is proved. □

We rewrite (6.19) in the form

$$\lambda^* = \max_{x\in V} \xi^*(x)\lambda^*\mathcal{G}_V^{(x,J)}(\lambda^*; x, x). \tag{6.20}$$

Let $\{D_V\}$ be an arbitrary (nonrandom) sequence of constants such that

$$D_V = l_V + \mathrm{O}(l_V^{-1}). \tag{6.21}$$

By $\zeta(x)$ we denote the expression under the maximum in (6.20) with $\lambda^*$ replaced by $D_V$ (6.21), i.e.,

$$\zeta(x) := \zeta(x; D_V) := \xi^*(x)D_V\mathcal{G}_V^{(x,J)}(D_V; x, x) \quad (x \in V).$$

We are interested in the limit theorem for $\zeta_{1,V}$ instead of $\lambda_{1,V}^{(J)}$. Let us introduce the following system of equations (cf. (6.10) and (6.11)):

$$AQ(\widetilde{T}_U; D_V; \widetilde{B}) - \log|V| + \beta\log l_V + \left(\beta - \frac{\alpha}{2}\right)\sum_{x\in U}\log\widetilde{T}_x + C(\alpha) = 0 \tag{6.22}$$

and

$$Q^{(x)}(\widetilde{T}_U; D_V; \widetilde{B}) = 0 \quad \text{for any } x \in U. \tag{6.23}$$

Obviously the solutions $\widetilde{B} = \widetilde{B}(D_V)$ and $\widetilde{T}_x = \widetilde{T}_x(D_V)$ satisfy (6.12) and (6.13), respectively.

**Lemma 6.7** *If* $\widetilde{B}(D_V)$ *is defined by* (6.22) *and* (6.23), *then*

$$\lim_V \mathbb{P}(\zeta_{1,V} \leqslant \widetilde{B}(D_V) + a_V^{-1}t) = \exp\{-\mathrm{e}^{-t}\} \quad \textit{for all } t \in \mathbb{R}.$$

Lemma 6.7 is proved below.

We now use Lemmas 6.6 and 6.7 to derive the asymptotic equations for $B_V$ appearing in (6.18). We first note that we may replace $\lambda^*$ on the right of (6.20) by an initial approximation

$D_V$ to obtain more explicit asymptotics for $\lambda^*$ on the left of (6.20). This enables us to apply the following scheme of iterations:

Write $B_V^{(0)} := l_V$, and for any $n \in \mathbb{N}$, let $B_V^{(n)} := \widetilde{B}(B_V^{(n-1)})$ be a solution of the system of equations (6.22) and (6.23) with $B_V^{(n-1)}$ replacing $D_V$. Now, since clearly $\lambda^* = B_V^{(0)} + \mathrm{O}(l_V^{-1})$ and $\max_{x\in V} \xi^*(x) = l_V + \mathrm{o}(1)$ in probability, we may rewrite the right-hand side of (6.20) in the form

$$\lambda^* = \max_{x\in V} \zeta(x;\, B_V^{(0)}) + \mathrm{O}(l_V^{-3}) \tag{6.24}$$

in probability. Applying Lemma 6.7 to the right-hand side of (6.24), we note that $\lambda^* = B_V^{(1)} + \mathrm{O}(a_V^{-1}) + \mathrm{O}(l_V^{-3})$ in probability. Substitute this into the right-hand side of (6.20) to get that

$$\lambda^* = \max_{x\in V} \zeta(x;\, B_V^{(1)}) + \mathrm{o}(a_V^{-1}) + \mathrm{O}(l_V^{-5})$$

in probability, and so on. Having repeated this procedure $J = [\alpha/2]$ times, we obtain the expression

$$\lambda^* = \max_{x\in V} \zeta(x;\, B_V^{(J-1)}) + \mathrm{o}(a_V^{-1})$$

(which in turn is equal to $B_V^{(J)} + \mathrm{O}(a_V^{-1})$ in probability). This and Lemma 6.7 imply (6.18) with $B_V := B_V^{(J)}$. Finally, substituting $D_V := B_V^{(J-1)} = B_V^{(J)} + \mathrm{O}(l_V^{-2J+1})$ into equations (6.22) and (6.23), by the straightforward calculations we obtain that $B_V$ satisfies equations (6.10) and (6.11) with $\mathrm{o}(a_V^{-1})$ accuracy, as claimed.

*Proof of Lemma 6.7* To this end, we abbreviate $\widetilde{B}_V := \widetilde{B}(D_V)$ and $U' := \{1 \leqslant |x| \leqslant J\} \subset V$. Fix $t \in \mathbb{R}$ arbitrarily. We need to show that

$$|V|\mathbb{P}\big(\zeta(0) \geqslant \widetilde{B}_V + a_V^{-1} t\big) \to \mathrm{e}^{-t}$$

(cf. Lemma 5.3) or, equivalently, that

$$p_V := |V| \int_{\mathbb{R}^{|U'|}} \left(1 - F\left(\frac{\widetilde{B}_V + a_V^{-1} t}{D_V g(s_{U'};\, D_V)}\right)\right) \prod_{x\in U'} p(s_x)\,\mathrm{d}s_x \to \mathrm{e}^{-t} \tag{6.25}$$

as $|V| \to \infty$, where the function $g(\cdot;\, D_V)$ is given by (6.6). We distinguish between three cases: ($\mathrm{i}_1$) $\alpha - 2[\alpha/2] > 1$, ($\mathrm{i}_2$) $\alpha - 2[\alpha/2] = 1$ and ($\mathrm{i}_3$) $\alpha - 2[\alpha/2] < 1$.

($\mathrm{i}_1$) Let $\alpha - 2[\alpha/2] > 1$. Then $J = j$, therefore, $U' = U$. Since $D_V g(\cdot;\, D_V) = 1 + \mathrm{o}(1)$ uniformly in $\mathbb{R}^{|U|}$ (Remark 6.1), we have that

$$p_V = \big(\mathrm{e}^{-t} + \mathrm{o}(1)\big)|V| l_V^{-\beta} \int_{\mathbb{R}^{|U|}} \exp\big\{-AQ(s_U;\, D_V;\, \widetilde{B}_V)\big\} \prod_{x\in U} \bar{p}(s_x)\,\mathrm{d}s_x, \tag{6.26}$$

where $\bar{p}(s) := p(s)\exp\{As_+^\alpha\}$ ($s \in \mathbb{R}$), and $Q(\cdot;\, D_V;\, \widetilde{B}_V)$ is defined by (6.7). To estimate the integral in (6.26), we introduce the rectangles

$$E := (-\infty, l_V]^{|U|} \quad \text{and} \quad E_\rho := (-\rho, \rho)^{|U|}$$

$(0 < \rho < 1)$, and the following function

$$W_V(t_U) := Q\big(\big\{(t_x + 1)\widetilde{T}_{x;V} : x \in U\big\};\, D_V;\, \widetilde{B}_V\big)$$

$$- Q\big(\{\widetilde{T}_{x;V} : x \in U\}; D_V; \widetilde{B}_V\big), \quad t_U \in E. \tag{6.27}$$

Now, making a change of integration variables $s_x := (t_x + 1)\widetilde{T}_{x;V}$ $(x \in U)$, we obtain that the right-hand side of (6.26) is equal to

$$p_V = \big(\mathrm{e}^{-t} + \mathrm{o}(1)\big)|V| l_V^{-\beta} \exp\big\{ - AQ(\widetilde{T}_{U;V}; D_V; \widetilde{B}_V)\big\}\bigg(\prod_{x\in U} \widetilde{T}_{x;V}\bigg) I_V + \mathrm{o}(1), \tag{6.28}$$

where

$$I_V := \int_E \mathrm{e}^{-AW_V(t_U)} \prod_{x\in U} \bar{p}\big((t_x + 1)\widetilde{T}_{x;V}\big)\, \mathrm{d}t_x. \tag{6.29}$$

To estimate $I_V$, we need some properties of $W_V(\cdot)$.

**Lemma 6.8** *The function $W_V(\cdot)$ is twice continuously differentiable in $E_1$ and possesses the following properties for any $x \in U$:*

1) $W_V(0) = 0$ *for any* $V$;
2) $W_V^{(x)}(0) = 0$ *for any* $V$;
3) $W_V^{(xx)}(0) = \alpha(\alpha - 1)\widetilde{T}_{x;V}^{\alpha}(1 + \mathrm{o}(1))$ *as* $|V| \to \infty$;
4) $W_V^{(xy)}(\cdot) = \mathrm{o}((W_V^{(xx)}(0) W_V^{(yy)}(0))^{1/2})$ *uniformly in* $E_1$ *as* $|V| \to \infty$, *for any* $y \in U \setminus \{x\}$;
5) $|W_V^{(xx)}(\cdot) - W_V^{(xx)}(0)| \leqslant C\rho W_V^{(xx)}(0)$ *in* $E_\rho$ *for any* $0 < \rho < \rho^0$ *and any* $V \supset V_0(\rho)$, *where* $C > 0$ *does not depend on* $V$ *and* $\rho$, *and moreover* $C\rho^0 < 1$;
6) $W_V(\cdot) \geqslant l_V^{\mathrm{const}(\rho)}$ *in* $E \setminus E_\rho$ *for some* $\mathrm{const}(\rho) > 0$ *and for any* $0 < \rho < 1$ *and any* $V \supset V_0(\rho)$.

*Proof* Properties 1 and 2 follow from (6.27) and (6.23), respectively.

In order to prove assertions 3–6, we need to estimate the derivatives of $g(\cdot; D_V)$ (6.8). Let us expand $g^{(xy)}(s_U; D_V)$ over $\kappa\Delta$ as in the case of the first derivatives (see Remark 6.1). In this expansion with $|V| \to \infty$, the main asymptotic contribution comes from the summands

$$C(\Gamma_{xy})(D_V - s_x)^{-1}(D_V - s_y)^{-1} \prod_{v\in V} (D_V - s_v)^{-n_v(\Gamma_{xy})}$$

corresponding to the paths $\Gamma_{xy}$ with $|\Gamma_{xy}| = |x|+|y|+|x-y|$ visiting both the sites $x$ and $y$; here $C(\Gamma_{xy}) > 0$. From this and Remark 6.1, it follows that uniformly in $(0, (2/3)^{1/\alpha} l_V)^{|U|}$

$$l_V g(\cdot; D_V) = 1 + \mathrm{o}(1), \qquad g^{(x)}(\cdot; D_V) \asymp l_V^{-2|x|-2} \tag{6.30}$$

and

$$g^{(xy)}(\cdot; D_V) \asymp l_V^{-|x|-|y|-|x-y|-3} \tag{6.31}$$

as $|V| \to \infty$ for any $x \in U$ and any $y \in U$.

We now pass from $g(s_U; D_V)$ through $Q(s_U; D_V; \widetilde{B}_V)$ (6.7) to $W_V(t_U)$ (6.27) with $t_x := -1 + s_x/\widetilde{T}_{x;V}$ $(x \in U)$. For $x \in U$ and $y \in U$, we compute the derivatives $W_V^{(xy)}(t_U)$ $(t_U \in E_1)$ and then use (6.13), (6.30) and (6.31) to obtain properties 3–5.

Let us show assertion 6. Again by (6.13), (6.30) and (6.31), we obtain that the quadratic form $\{W_V^{(xy)}(\cdot)\}_{x\in U, y\in U}$ is positively defined (therefore, $W_V(\cdot)$ is a convex function) in the rectangle $\bigtimes_{x\in U}[\delta - 1, (2/3)^{1/\alpha} l_V/\widetilde{T}_{x;V} - 1) \subset E$, for any $0 < \delta < 1$ and any $V \supset V_0(\delta)$. Further, for $x \in U$ chosen arbitrarily, let us consider the function $\widetilde{W}_V(t_x) := W_V(t_U)$, $t_x <$

$l_V$, with the remaining variables $t_y < l_V$ $(y \in U \setminus \{x\})$ fixed arbitrarily. By a straightforward calculation based on (6.13) and (6.30), we can find small $\delta > 0$ independent of $V$ and $t_U$ such that $\widetilde{W}_V(\cdot)$ is decreasing in $(-\infty, \delta - 1)$ and, therefore,

$$\widetilde{W}_V(\cdot) \geqslant \widetilde{W}_V(\delta - 1) \quad \text{in } \left(-\infty, \delta - 1\right] \cup \left[\left(\frac{2}{3}\right)^{1/\alpha} \frac{l_V}{\widetilde{T}_{x;V}} - 1, l_V\right);$$

$V \supset V_0(\delta)$. Summarizing these properties of $W_V(\cdot)$, we obtain that

$$W_V(\cdot) \geqslant \inf_{s_U \in \partial E_\rho} W_V(s_U) \quad \text{in } E \setminus E_\rho \tag{6.32}$$

$(0 < \rho < 1;\ V \supset V_0(\rho))$; here $\partial E_\rho$ stands for the boundary of the cube $E_\rho \subset \mathbb{R}^{|U|}$. We then expand the function under the infimum in powers of $s_x$ $(x \in U)$ and use properties 1–5 to see that the right-hand side of (6.32) is bigger than $\mathrm{const}\rho^2 \min_{x \in U} \widetilde{T}^{\alpha}_{x;V}$.This and (6.13) imply assertion 6. Lemma 6.8 is proved. □

From Lemma 6.8 we see that the function $W_V(\cdot)$ attains at $0 \in \mathbb{R}^{|U|}$ its global minimum of elliptic type; $V \supset V_0$. We may, therefore, apply Laplace's method to estimate $I_V$ (6.29). Indeed, let us split $I_V = \int_{E_\rho} + \int_{E \setminus E_\rho}$. By assertion 6 of Lemma 6.8 the second integral does not exceed $\mathrm{O}(\exp\{-l_V^{\mathrm{const}}\})$ as $|V| \to \infty$ for each $0 < \rho < 1$. As for the first integral, we apply Taylor's formula to expand $W_V(t_U)$ $(t_U \in E_\rho)$ in powers of $t_x$ $(x \in U)$ up to the quadratic form, and then use assertions 1–5 of Lemma 6.8. Thus, a straightforward calculation (recall $\bar{p}(t) = A\alpha t^{\alpha-\beta-1}(1 + \mathrm{o}(1))$) shows that

$$I_V = \chi(V; \rho)\left(\frac{2\pi A\alpha}{\alpha - 1}\right)^{|U|/2} \prod_{x \in U} \widetilde{T}_{x;V}^{\frac{\alpha}{2}-\beta-1} + \mathrm{O}\big(\exp\{-l_V^{\mathrm{const}}\}\big),$$

where $\chi(V; \rho) \to 1$ letting first $|V| \to \infty$ and afterwards $\rho \to 0$. This and (6.28) combined with (6.22) imply (6.25) for $\alpha - 2[\alpha/2] > 1$, as required.

($\mathrm{i}_2$) Assume now that $\alpha - 2[\alpha/2] = 1$, or equivalently, $\alpha = 2n + 1$ $(n \in \mathbb{N})$. Then $J = j + 1$, therefore, $U' \supset U$. Abbreviate $U'' := U' \setminus U = \{x : |x| = [\alpha/2]\}$. We restrict ourselves to the proof for $\alpha \geqslant 5$ (the case $\alpha = 3$ is similar). Split $p_V = p_{V,M} + q_{V,M}$, where $p_{V,M}$ (resp., $q_{V,M}$) stands for the left-hand side of (6.25) with the integration domain $E^{(M)} := \{s_{U'} \colon s_U \in \mathbb{R}^{|U|}, s_{U''} \in (-M, M)^{|U''|}\}$ (resp., $\mathbb{R}^{|U'|} \setminus E^{(M)}$) instead of $\mathbb{R}^{|U'|}$; here $M > 0$.

We first consider $p_{V,M}$. For arbitrarily fixed $s_U \in \mathbb{R}^{|U|}$, let us expand the function $g(s_{U'}; D_V)^{-\alpha}$ $(s_{U''} \in (-M, M)^{|U''|})$ in powers of $s_y$ $(y \in U'')$. Therefore, by (6.30) and (6.31) (see also Remark 6.1) we have that, as $|V| \to \infty$,

$$\begin{aligned} &g(s_{U'}; D_V)^{-\alpha} \\ &\quad = g(s_U; D_V)^{-\alpha} - \alpha g(s_U; D_V)^{-\alpha-1} \sum_{y \in U''} s_y \big(g(s_{U'}; D_V)\big)^{(y)}\Big|_{s_{U''}=0} + \mathrm{o}(1) \end{aligned} \tag{6.33}$$

uniformly in $E^{(M)}$ for arbitrarily fixed $M > 0$. In addition,

$$g(s_U; D_V)^{-\alpha-1}\big(g(s_{U'}; D_V)\big)^{(y)}\Big|_{s_{U''}=0} = \kappa^{\alpha-1} c_y + \mathrm{o}\,(1) \tag{6.34}$$

uniformly in $s_U \in (0, \mathrm{o}(l_V))^{|U|}$ for each $y \in U''$. Following the same lines of the proof in part ($\mathrm{i}_1$) combined with (6.33) and (6.34), we obtain that

$$\begin{aligned} p_{V,M} &= \left(\mathrm{e}^{-t} + \mathrm{o}(1)\right) \prod_{y\in U''} \mathbb{E}\big(\exp\big\{A\alpha\kappa^{\alpha-1}c_y\xi(0)\big\}\mathbb{1}\big\{|\xi(0)| < M\big\}\big) \\ &\quad \times l_V^{-\beta}|V| \int_{(0,\varepsilon l_V)^{|U|}} \exp\{-AQ(s_U; D_V; \widetilde{B}_V)\} \prod_{x\in U} \bar{p}(s_x)\, \mathrm{d}s_x + \mathrm{o}(1) \\ &= \mathrm{e}^{-t} + \mathrm{o}(1) \end{aligned}$$

($\varepsilon > 0$) passing to the limits first $|V| \to \infty$ and then $M \to \infty$. Similarly, noting that

$$g(s_{U'}; D_V)^{-\alpha} \geqslant g(s_U; D_V)^{-\alpha} - \text{const} \sum_{y\in U''} (s_y)_+$$

for each $s_{U'} \in \mathbb{R}^{|U'|}$ and each $V \supset V_0$, we obtain that

$$\begin{aligned} q_{V,M} &\leqslant \text{const}'\big(\mathbb{E}\big(\exp\{\text{const}\,\xi(0)\}\mathbb{1}\big\{\xi(0) \geqslant M\big\}\big) + \mathbb{P}\big(\xi(0) \leqslant -M\big)\big) \\ &\quad \times l_V^{-\beta}|V| \int_{\mathbb{R}^{|U|}} \exp\big\{-AQ(s_U; D_V; \widetilde{B}_V)\big\} \prod_{x\in U} \bar{p}(s_x)\, \mathrm{d}s_x \to 0, \end{aligned}$$

passing to the limits first $|V| \to \infty$ and then $M \to \infty$. This concludes the proof of (6.25) for $\alpha - 2[\alpha/2] = 1$.

($\mathrm{i}_3$) Assume that $\alpha - 2[\alpha/2] < 1$. Then $J = j + 1$, therefore, $U' \supset U$. Abbreviate $U'' := U' \setminus U = \{x : |x| = [\alpha/2]\}$. For fixed $x_U \in \mathbb{R}^{|U|}$, we again expand $g(s_{U'}; D_V)^{-\alpha}$ ($s_{U''} \in (-M, M)^{|U''|}$) in powers of $s_y$ ($y \in U''$). Therefore, we have that, as $|V| \to \infty$,

$$g(s_{U'}; D_V)^{-\alpha} = g(s_U; D_V)^{-\alpha} + \mathrm{o}(1) \tag{6.35}$$

uniformly in $E^{(M)} := \mathbb{R}^{|U|} \times (-M, M)^{|U''|}$ for fixed $M > 0$. The proof of (6.25) repeats from line to line that in part ($i_2$), where instead of (6.33) and (6.34) we use (6.35) and where all the factors $c_y$ ($y \in U''$) must be replaced by zero.

Lemma 6.7 is proved. □

## Appendix A: The Path Expansion for Resolvents

Let $V \subseteq \mathbb{Z}^\nu$ denote either the $\nu$-dimensional torus or the whole lattice $\mathbb{Z}^\nu$. Fix some subset $\Pi \subset V$, $0 < |\Pi| < \infty$. Given a realization $\{\xi(x) : x \in V\}$, let $\mathcal{G}(\lambda; x, y)$, $\widetilde{\mathcal{G}}(\lambda; x, y)$ and $\widetilde{\mathcal{G}}^{(u)}(\lambda; x, y)$ ($x \in V, y \in V$) denote Green's functions of the Hamiltonians $\mathcal{H} := \kappa\Delta + \xi(\cdot)$, $\widetilde{\mathcal{H}} := \kappa\Delta + \sum_{x\in V\setminus\Pi} \xi(x)\delta_x$ and $\widetilde{\mathcal{H}}^{(u)} := \widetilde{\mathcal{H}} + \xi(u)\delta_u$ in $l^2(V)$, respectively; $u \in \Pi$. Here and in the sequel, we suppress $V$ from all the notation.

**Lemma A.1** (Cluster expansion) *For all $x \in V$ and all $y \in V$,*

$$\begin{aligned} \mathcal{G}(\lambda; x, y) &= \widetilde{\mathcal{G}}(\lambda; x, y) \\ &\quad + \sum_{k\in N} \sum_{\gamma: u_1 \to u_2 \to \ldots \to u_k} \widetilde{\mathcal{G}}^{(u_1)}(\lambda; x, u_1)\xi(u_1)\left(\prod_{l=2}^{k} \widetilde{\mathcal{G}}^{(u_l)}(\lambda; u_{l-1}, u_l)\xi(u_l)\right)\widetilde{\mathcal{G}}(\lambda; u_k, y), \end{aligned} \tag{A.1}$$

*provided the series converges. Here the second sum* $\sum_\gamma$ *is taken over all paths*

$$\gamma : u_1 \to u_2 \to \ldots \to u_k \quad \textit{in } \Pi$$

*such that* $u_{i-1} \neq u_i$ *for each* $2 \leqslant i \leqslant k$, *having the length* $|\gamma| = k-1$; $\prod_{l=2}^{1} \ldots := 1$.

*Proof* Formula (A.1) has been announced by Golitsyna and Molchanov [26]. We show (A.1) for a completeness of our paper only.

The proof is by induction in $K \in \mathbb{N}$. Let us represent $\mathcal{G}(\lambda; \cdot, y) := \mathcal{G}(\lambda)\delta_y$ in the form

$$\mathcal{G}(\lambda)\delta_y = \widetilde{\mathcal{G}}(\lambda)\delta_y + e_1. \tag{A.2}$$

Then, applying the operator $\lambda - \mathcal{H} = \lambda - \widetilde{\mathcal{H}} - \sum_{u\in\Pi} \xi(u)\delta_u$ to both the sides of (A.2), we obtain the equation for $e_1 := e_1(\cdot)$

$$(\lambda - \mathcal{H})e_1 = \sum_{u\in\Pi} \delta_u \xi(u)\widetilde{\mathcal{G}}(\lambda; u, y).$$

For any $k \in \mathbb{N}$, let us write

$$\widetilde{e}_k(u_1) := \begin{cases} \displaystyle\sum_{\substack{\gamma:u_2\to\ldots\to u_k \\ u_2\neq u_1}} \left(\prod_{l=2}^{k} \widetilde{\mathcal{G}}^{(u_l)}(\lambda; u_{l-1}, u_l)\xi(u_l)\right)\widetilde{\mathcal{G}}(\lambda; u_k, y) & \text{if } k \geqslant 2, \\ \widetilde{\mathcal{G}}(\lambda; u_1, y) & \text{if } k = 1 \end{cases} \tag{A.3}$$

for all $u_1 \in \Pi$. Fix now $K \in \mathbb{N}\backslash\{1\}$ and assume that the following identity

$$\mathcal{G}(\lambda)\delta_y = \widetilde{\mathcal{G}}(\lambda)\delta_y + \sum_{k=1}^{K-1} \sum_{u\in\Pi} \widetilde{\mathcal{G}}^{(u)}(\lambda)\delta_u \xi(u)\widetilde{e}_k(u) + e_K \tag{A.4}$$

holds with $e_K := e_K(\cdot)$ satisfying the equation

$$(\lambda - \mathcal{H})e_K = \sum_{u\in\Pi} \delta_u \xi(u)\widetilde{e}_K(u). \tag{A.5}$$

Let us show (A.4) and (A.5) with $K + 1$ replacing $K$. Put $e_K$ in the form

$$e_K = \sum_{u\in\Pi} e_{K,u} + e_{K+1}, \tag{A.6}$$

where $e_{K,u} = e_{K,u}(\cdot)$ satisfies the equation

$$(\lambda - \widetilde{\mathcal{H}}^{(u)})e_{K,u} = \delta_u \xi(u)\widetilde{e}_K(u). \tag{A.7}$$

Thus, $e_{K,u}$ can be written in the form

$$e_{K,u}(\cdot) = \widetilde{\mathcal{G}}^{(u)}(\lambda; \cdot, u)\xi(u)\widetilde{e}_K(u) \tag{A.8}$$

for any $u \in \Pi$. We now apply the operator $\lambda - \mathcal{H} = \lambda - \widetilde{\mathcal{H}}^{(u)} - \sum_{v\in\Pi\backslash\{u\}} \delta_v \xi(v)$ $(u \in \Pi)$ to both the sides of (A.6). Then

$$(\lambda - \mathcal{H})e_K = \sum_{u\in\Pi} (\lambda - \mathcal{H})e_{K,u} + (\lambda - \mathcal{H})e_{K+1}$$

$$= \sum_{u\in\Pi}(\lambda - \widetilde{\mathcal{H}}^{(u)})e_{K,u} - \sum_{u\in\Pi}\sum_{v\in\Pi\setminus\{u\}} \delta_v\xi(v)e_{K,u}(v) + (\lambda - \mathcal{H})e_{K+1}$$

$$= \sum_{u\in\Pi}\delta_u\xi(u)\widetilde{e}_K(u) - \sum_{u\in\Pi}\sum_{v\in\Pi\setminus\{u\}} \delta_u\xi(u)e_{K,v}(u) + (\lambda - \mathcal{H})e_{K+1}$$

by (A.7). Combining this with (A.5) and (A.8) and noting that (by the definition of $\widetilde{e}_k$ in (A.3)) $\widetilde{e}_{K+1}(u) = \sum_{v\in\Pi\setminus\{u\}}\widetilde{\mathcal{G}}^{(v)}(\lambda; u, v)\xi(v)\widetilde{e}_K(v)$ for any $u \in \Pi$, we obtain the equation for $e_{K+1} := e_{K+1}(\cdot)$

$$(\lambda - \mathcal{H})e_{K+1} = \sum_{u\in\Pi}\delta_u\xi(u)\widetilde{e}_{K+1}(u). \tag{A.9}$$

Finally, summarizing (A.8), (A.6) and (A.4), we get that

$$\mathcal{G}(\lambda)\delta_y = \widetilde{\mathcal{G}}(\lambda)\delta_y + \sum_{k=1}^{K}\sum_{u\in\Pi}\widetilde{\mathcal{G}}^{(u)}(\lambda)\delta_u\xi(u)\widetilde{e}_k(u) + e_{K+1},$$

where $e_{K+1} := e_{K+1}(\cdot)$ satisfies equation (A.9). These inductional arguments prove (A.4) and (A.5) for any $K \in \mathbb{N}$. □

The following result is well known (see, e.g., Appendix A in [18] or p. 302 in [38]).

**Lemma A.2** (Expansion over $\kappa\Delta$) *For any $x \in V$ and any $y \in V$,*

$$\mathcal{G}(\lambda; x, y) = \sum_{\Gamma(x,y)}\kappa^{|\Gamma(x,y)|}\prod_{v\in V}(\lambda - \xi(v))^{-n_v(\Gamma(x,y))}, \tag{A.10}$$

*provided the series converges. Here the sum $\sum_{\Gamma(x,y)}$ is taken over all paths*

$$\Gamma : v_0 := x \to v_1 \to \ldots \to v_m := y \quad \text{in } V$$

*such that $|v_i - v_{i-1}| = 1$ for each $1 \leqslant i \leqslant m$ and each $m \in \mathbb{N}$ (i.e., the nearest neighbor paths in $V$ starting at $x$ and ending at $y$); $n_v(\Gamma(x, y))$ denotes the number of times the path $\Gamma(x, y)$ visits the site $v \in V$; $|\Gamma(x, y)| := \sum_{v\in V} n_v(\Gamma(x, y)) - 1 \geqslant |x - y|$ (note that if $y = x$ and $|\Gamma(x, x)| = 0$, then the corresponding summand in (A.10) is equal to $(\lambda - \xi(x))^{-1}$).*

The proof of (A.10) is trivial. Indeed, since, for fixed $y \in V$, the function $\mathcal{G}(\lambda; \cdot, y)$ satisfies the equation

$$\lambda\mathcal{G}(\lambda; \cdot, y) - \kappa\Delta\mathcal{G}(\lambda; \cdot, y) - \xi(\cdot)\mathcal{G}(\lambda; \cdot, y) = \delta_y(\cdot),$$

result (A.10) is done by applying the following iteration formula

$$\mathcal{G}(\lambda; x, y) = \frac{\kappa\sum_{|x'-x|=1}\mathcal{G}(\lambda; x', y)}{\lambda - \xi(x)} + \frac{\delta_y(x)}{\lambda - \xi(x)}, \quad x \in V.$$

## Appendix B: Upper Part of Spectrum in Deterministic Rare Scatterers Model

### B.1 Spectral Problem

We again consider the Hamiltonian $\mathcal{H} = \kappa\Delta + \xi(\cdot)$ in $l^2(V)$, where $V \subseteq \mathbb{Z}^\nu$ denotes either the $\nu$-dimensional (finite) torus or the whole lattice $\mathbb{Z}^\nu$. In this subsection, we reduce the spectral problem

$$\mathcal{H}\psi(\cdot) = \lambda\psi(\cdot); \quad \psi(\cdot) \in l^2(V), \quad \lambda \in \mathbb{R}, \tag{B.1}$$

to a certain dispersion equation for an eigenvalue $\lambda$, provided $\lambda$ satisfies appropriate conditions depending upon the structure of potential $\xi(\cdot)$ in $V$ (Theorem B.1).

Let us introduce some abbreviations we use throughout Sect. B. (To this end, we suppress $V$ from all the notation.) Fix a constant $0 < L < \infty$, and define the subset $\Pi \subset V$ by

$$\Pi := \{x \in V : \xi(x) \geqslant L\}. \tag{B.2}$$

We throughout assume that

$$0 < |\Pi| < \infty.$$

Write

$$\widetilde{\xi}(x) := \begin{cases} \xi(x) & \text{if } x \in V \backslash \Pi, \\ 0 & \text{otherwise.} \end{cases} \tag{B.3}$$

Let $\mathcal{G}^{(z)}(\lambda; x, y)$, $\widetilde{\mathcal{G}}(\lambda; x, y)$ and $\widetilde{\mathcal{G}}^{(z)}(\lambda; x, y)$ $(x \in V, y \in V)$ be Green's functions of the Hamiltonians $\mathcal{H}^{(z)} := \kappa\Delta + (1 - \delta_z)\xi(\cdot)$, $\widetilde{\mathcal{H}} := \kappa\Delta + \widetilde{\xi}(\cdot)$ and $\widetilde{\mathcal{H}}^{(z)} := \widetilde{\mathcal{H}} + \xi(z)\delta_z$ in $l^2(V)$, respectively; $z \in \Pi$. Fix constant $\lambda_0 > L + 2\nu\kappa$. For any $\lambda \geqslant \lambda_0$, we write

$$A(\lambda) := \log \frac{\lambda - L}{2\nu\kappa} \tag{B.4}$$

and

$$B(\lambda; u) := b(\lambda)\lambda^{-2} \left| \frac{1}{\xi(u)} - \widetilde{\mathcal{G}}(\lambda; u, u) \right|^{-1}, \quad \text{where } b(\lambda) := \frac{(\lambda - L)^2}{\lambda - L - 2\nu\kappa}.$$

Abbreviate

$$r := \min\bigl\{|x - y| : x \in \Pi, y \in \Pi, x \neq y\bigr\} \quad \text{if } |\Pi| \geqslant 2, \tag{B.5}$$

and $r := |V|^{1/\nu}$ if $|\Pi| = 1$, by convention.

For $z \in \Pi$, let us introduce the following (close) subsets $\Lambda(z) \subset [\lambda_0, \infty)$:

$$\begin{aligned} \Lambda(z) &:= \Lambda(z; \lambda_0; \delta; \xi(\cdot)) \\ &:= \left\{ \lambda \geqslant \lambda_0 : \max_{u \in \Pi \backslash \{z\}} B(\lambda; u) \leqslant \frac{1}{2|\Pi|} \mathrm{e}^{\delta A(\lambda) r} \right\} \quad \text{if } |\Pi| \geqslant 2, \end{aligned} \tag{B.6}$$

and $\Lambda(z) := [\lambda_0, \infty)$ if $|\Pi| = 1$, by convention; here $0 < \delta < 1$.

**Theorem B.1** *Fix $z \in \Pi$ and $0 < \delta < 1$. Then the following assertions hold.*

(i) $\operatorname{Spect}(\mathcal{H}^{(z)}) \cap \Lambda(z) = \varnothing$.
(ii) *If $\lambda$ belongs to $\Lambda(z)$, then $\lambda$ is an eigenvalue of $\mathcal{H}$ if and only if $\lambda$ satisfies the equation*

$$\mathcal{G}^{(z)}(\lambda; z, z) = \frac{1}{\xi(z)}. \tag{B.7}$$

*In this case, the corresponding (normalized) eigenfunction has the form*

$$\psi(\cdot;\lambda) = \mathcal{G}^{(z)}(\lambda;\cdot,z)\Bigg(\sum_{y\in V}\big(\mathcal{G}^{(z)}(\lambda;y,z)\big)^2\Bigg)^{-1/2}. \tag{B.8}$$

(iii) *For any* $\lambda \in \Lambda(z)$,

$$\big|\mathcal{G}^{(z)}(\lambda;x,z)\big| \leqslant \frac{2b(\lambda)}{\lambda(\lambda-L)}\,\mathrm{e}^{-(1-\delta)A(\lambda)|x-z|} \quad \textit{for all } x\in V, \tag{B.9}$$

*and*

$$\big|\mathcal{G}^{(z)}(\lambda;z,z) - \widetilde{\mathcal{G}}(\lambda;z,z)\big| \leqslant \frac{b(\lambda)}{\lambda^2}\,\mathrm{e}^{-(2-\delta)A(\lambda)r}. \tag{B.10}$$

Let us first explain the main idea of the proof of Theorem B.1. We focus on the cluster expansion formula (A.1) with $\Pi$ defined by (B.2) and with $\mathcal{G}(\lambda)$ replaced by $\mathcal{G}^{(z)}(\lambda)$. Since, for each $\lambda\in\Lambda(z)$, Green's functions $\widetilde{\mathcal{G}}(\lambda;\cdot,\cdot)$ and $\widetilde{\mathcal{G}}^{(u)}(\lambda;\cdot,\cdot)$ of the Hamiltonians $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}^{(u)}$ ($u\in\Pi\backslash\{z\}$) are shown to decay exponentially fast, one finds that the series (A.1) converges. Consequently, $\Lambda(z)$ is contained in the resolvent set of the Hamiltonian $\mathcal{H}^{(z)}$. This in turn enables us to reduce the spectral problem (B.1) to the dispersion equation (B.7) for eigenvalues in $\Lambda(z)$.

*Proof of Theorem B.1* We shall treat the case where $|\Pi|\geqslant 3$. For $|\Pi|=1$, the proof is trivial. For $|\Pi|=2$, the proof is similar (and even simpler) as in the case $|\Pi|\geqslant 3$.

In the sequel, we need the following estimates for the functions $\widetilde{\mathcal{G}}(\lambda;\cdot,u)$ and $\widetilde{\mathcal{G}}^{(u)}(\lambda;\cdot,u)$ with $u\in\Pi$.

**Lemma B.2** *Fix* $u\in\Pi$, $v\in\Pi\backslash\{u\}$ *and* $\lambda > L+2\nu\kappa$ *such that* $\xi(u)\widetilde{\mathcal{G}}(\lambda;u,u)\neq 1$. *Then, for all* $x\in V$,

$$\big|\widetilde{\mathcal{G}}(\lambda;x,u)\big| \leqslant \frac{b(\lambda)}{\lambda(\lambda-L)}\,\mathrm{e}^{-A(\lambda)|x-u|}, \tag{B.11}$$

$$\big|\widetilde{\mathcal{G}}(\lambda;v,u)\big| \leqslant \frac{b(\lambda)}{\lambda^2}\,\mathrm{e}^{-A(\lambda)|v-u|}, \tag{B.12}$$

$$\big|\widetilde{\mathcal{G}}^{(u)}(\lambda;x,u)\big| \leqslant \frac{\lambda B(\lambda;u)}{(\lambda-L)\xi(u)}\,\mathrm{e}^{-A(\lambda)|x-u|}, \tag{B.13}$$

*and*

$$\big|\widetilde{\mathcal{G}}^{(u)}(\lambda;v,u)\big| \leqslant \frac{B(\lambda;u)}{\xi(u)}\,\mathrm{e}^{-A(\lambda)|v-u|}. \tag{B.14}$$

*Proof* According to Lemma A.1 with $\{u\}$ instead of $\Pi$ and with $\widetilde{\mathcal{G}}^{(u)}(\lambda)$ instead of $\mathcal{G}(\lambda)$, we obtain the following resolvent identity

$$\widetilde{\mathcal{G}}^{(u)}(\lambda;x,u) = \widetilde{\mathcal{G}}(\lambda;x,u) + \widetilde{\mathcal{G}}^{(u)}(\lambda;x,u)\xi(u)\widetilde{\mathcal{G}}(\lambda;u,u).$$

Then

$$\widetilde{\mathcal{G}}^{(u)}(\lambda;x,u) = \widetilde{\mathcal{G}}(\lambda;x,u)\big/\big(1-\xi(u)\widetilde{\mathcal{G}}(\lambda;u,u)\big).$$

We now apply Lemma A.2 to expand $\widetilde{\mathcal{G}}(\lambda; x, u)$ over $\kappa\Delta$. Using this expansion combined with the fact that the total number of paths $\Gamma(x, u)$ of the length $k$ does not exceed $(2\nu)^{k-1}$, it is easy to obtain the claimed estimates. □

We now turn to the proof of Theorem B.1.

(i) For fixed $y \in V$ and $\lambda \in \Lambda(z)$, let us consider the equation

$$(\lambda - \mathcal{H}^{(z)})\varphi(\cdot) = \sum_{u \in \Pi\backslash\{z\}} \delta_u(\cdot)\xi(u)\widetilde{\mathcal{G}}(\lambda; u, y). \tag{B.15}$$

Having written $\lambda - \mathcal{H}^{(z)} = \lambda - \widetilde{\mathcal{H}} - \sum_{u \in \Pi\backslash\{z\}} \xi(u)\delta_u$, we apply the operator $\widetilde{\mathcal{G}}(\lambda) := (\lambda - \widetilde{\mathcal{H}})^{-1}$ to both the sides of (B.15) to represent (B.15) in the following form

$$\varphi(\cdot) - \sum_{u \in \Pi\backslash\{z\}} \widetilde{\mathcal{G}}(\lambda; \cdot, u)\xi(u)\varphi(u) = \sum_{u \in \Pi\backslash\{z\}} \widetilde{\mathcal{G}}(\lambda; \cdot, u)\xi(u)\widetilde{\mathcal{G}}(\lambda; u, y). \tag{B.16}$$

According to Gerzhgorin's theorem (e.g., Theorem 7.2.1 in [32]), equation (B.16) has an unique solution $\varphi(\cdot)$, provided that

$$\left|\widetilde{\mathcal{G}}(\lambda; u, u) - \frac{1}{\xi(u)}\right| > \sum_{v \in \Pi\backslash\{z,u\}} \widetilde{\mathcal{G}}(\lambda; v, u) \quad \text{for any } u \in \Pi\backslash\{z\}. \tag{B.17}$$

On the other hand, it is easy to show (B.17) by applying (B.12) to the right-hand side of (B.17) combined with the definition of $\lambda \in \Lambda(z)$.

We now put

$$w(\cdot) := \widetilde{\mathcal{G}}(\lambda; \cdot, y) + \varphi(\cdot), \tag{B.18}$$

where $\varphi(\cdot)$ is the solution of equation (B.15). We then apply the Hamiltonian $\lambda - \mathcal{H}^{(z)}$ to both the sides of (B.18) to get the identity $w(\cdot) \equiv \mathcal{G}^{(z)}(\lambda; \cdot, y)$. Since $y \in V$ is chosen arbitrarily, this implies that $\lambda$ does not belong to $\mathrm{Spect}(\mathcal{H}^{(z)})$. Part (i) is proved.

Part (ii) follows from (i) by considering $\mathcal{H}^{(z)}$ as a "basic" operator in the representation $\mathcal{H} = \mathcal{H}^{(z)} + \xi(z)\delta_z$ [27]. Indeed, let (B.1) be fulfilled for some $\lambda \in \Lambda(z)$ and for some $\psi(\cdot) \not\equiv 0$. Rewrite (B.1) in the form $(\lambda - \mathcal{H}^{(z)})\psi(\cdot) = \xi(z)\psi(z)\delta_z(\cdot)$, and then apply the resolvent operator $\mathcal{G}^{(z)}(\lambda) := (\lambda - \mathcal{H}^{(z)})^{-1}$. (Recall that $\xi(z) > 0$ by the assumption.) It is easy to see that $\lambda$ satisfies (B.7), and the corresponding (normalized) eigenfunction is given by (B.8). The converse follows by the same calculations.

(iii) We first note that part (i) implies $\left|\mathcal{G}^{(z)}(\lambda; x, y)\right| < \infty$ and that (B.13) yields $\left|\widetilde{\mathcal{G}}^{(u)}(\lambda; x, u)\right| < \infty$ for any $\lambda \in \Lambda(z)$, any $u \in \Pi\backslash\{z\}$ and all $x \in V$, $y \in V$. We may therefore apply the cluster expansion for resolvents like in (A.1), where $\mathcal{G}(\lambda; \cdot, \cdot)$ is replaced by $\mathcal{G}^{(z)}(\lambda; \cdot, \cdot)$ and $\Pi$ is replaced by $\Pi\backslash\{z\}$. Thus, for each $\lambda \in \Lambda(z)$, each $x \in V$ and each $y \in V$,

$$\begin{aligned}\mathcal{G}^{(z)}(\lambda; x, y) = \widetilde{\mathcal{G}}(\lambda; x, y) &+ \sum_{k=1}^{K-1} \sum_{\gamma: u_1 \to u_2 \to \ldots \to u_k} \widetilde{\mathcal{G}}^{(u_1)}(\lambda; x, u_1)\xi(u_1) \\ &\times \left(\prod_{l=2}^{k} \widetilde{\mathcal{G}}^{(u_l)}(\lambda; u_{l-1}, u_l)\xi(u_l)\right)\widetilde{\mathcal{G}}(\lambda; u_k, y) + \rho_{\lambda,K}(x, y),\end{aligned} \tag{B.19}$$

where

$$\rho_{\lambda,K}(x,y) := \sum_{\gamma:u_1\to u_2\to\ldots\to u_K} \mathcal{G}^{(z)}(\lambda;x,u_1)\xi(u_1) \times \left(\prod_{l=2}^{K} \widetilde{\mathcal{G}}^{(u_l)}(\lambda;u_{l-1},u_l)\xi(u_l)\right)\widetilde{\mathcal{G}}(\lambda;u_K,y).$$

First, by (B.11), (B.14) and (B.6) we have that

$$\begin{aligned}\left|\rho_{\lambda,K}(x,y)\right| &\leqslant \mathrm{const}(\lambda;\xi(\cdot))\left(\mathrm{e}^{-A(\lambda)r}\max_{u\in\Pi\setminus\{z\}} B(\lambda;u)\right)^{K-1}|\Pi|^K \\ &\leqslant \mathrm{const}'(\lambda;\xi(\cdot))\left(\frac{1}{2}\right)^{K-1} \to 0 \quad \text{as } K\to\infty,\end{aligned} \tag{B.20}$$

where const $>0$ and const$'>0$ do not depend on $K$. We now apply (B.11)–(B.14) to estimate the $k$th summands in (B.19). Namely, from (B.19) with $y=z$ and $K\to\infty$ we have that

$$\begin{aligned}&\left|\mathcal{G}^{(z)}(\lambda;x,z)-\widetilde{\mathcal{G}}(\lambda;x,z)\right| \\ &\quad\leqslant \frac{b(\lambda)}{\lambda(\lambda-L)}\sum_{k\in\mathbb{N}}\sum_{\gamma:u_1\to u_2\to\ldots\to u_k} B(\lambda;u_1)\,\mathrm{e}^{-A(\lambda)|x-u_1|} \\ &\qquad\times\left(\prod_{l=2}^{k} B(\lambda;u_l)\,\mathrm{e}^{-A(\lambda)|u_{l-1}-u_l|}\right)\mathrm{e}^{-A(\lambda)|u_k-z|} \\ &\quad\leqslant \frac{b(\lambda)}{\lambda(\lambda-L)}\,\mathrm{e}^{-(1-\delta)A(\lambda)|x-z|}\sum_{k\in\mathbb{N}}\left(|\Pi|\max_{u\in\Pi\setminus\{z\}} B(\lambda;u)\,\mathrm{e}^{-\delta A(\lambda)r}\right)^k \\ &\quad\leqslant \frac{b(\lambda)}{\lambda(\lambda-L)}\,\mathrm{e}^{-(1-\delta)A(\lambda)|x-z|}\end{aligned} \tag{B.21}$$

by using the definition of $\lambda\in\Lambda(z)$ as in (B.6). This combined with (B.11) gives (B.9). Proceeding as in (B.20) and (B.21), we also obtain (B.10). Theorem B.1 is proved. □

### B.2 Exact Estimates for Extreme Eigenvalues and Eigenfunctions

We now are in a position to study the structure of the boundary part of Spect($\mathcal{H}$), provided the potential satisfies conditions like those in (2.5)–(2.8) (see Theorem B.3 below). As before, the Hamiltonian $\mathcal{H}=\kappa\Delta+\xi(\cdot)$ acts on $l^2(V)$, where $V\subseteq\mathbb{Z}^\nu$ is either the $\nu$-dimensional (finite) torus or the whole lattice $\mathbb{Z}^\nu$.

Let us introduce the following notation. As in Sect. B.1, let $\Pi\subset V$ stand for the subset of $\xi_V$-exceedances of the level $0<L<\infty$ defined by (B.2) and $\widetilde{\xi}(\cdot)$ for the "noise" potential defined by (B.3). For any $u\in\Pi$, by $\widetilde{\lambda}(u)$ we abbreviate the principal eigenvalue of the "single peak" Hamiltonian $\widetilde{\mathcal{H}}^{(u)}:=\kappa\Delta+\widetilde{\xi}(\cdot)+\xi(u)\delta_u$ in $l^2(V)$. Given $0<h<\infty$, we introduce the subset

$$\widetilde{\Pi}:=\{u\in\Pi:\widetilde{\lambda}(u)\geqslant L+2\nu\kappa+h\}.$$

If $\widetilde{\Pi} \neq \varnothing$, let

$$\widetilde{\lambda}_1 \geqslant \widetilde{\lambda}_2 \geqslant \cdots \geqslant \widetilde{\lambda}_{|\widetilde{\Pi}|} \tag{B.22}$$

be the variational series of the sample $\{\widetilde{\lambda}(x) : x \in \widetilde{\Pi}\}$, and write $\widetilde{\lambda}_{|\widetilde{\Pi}|+1} := L + 2\nu\kappa + h$. Let $A(\lambda)$ be given by (B.4), i.e. $A(\lambda) := \log \frac{\lambda - L}{2\nu\kappa}$ for each $\lambda \geqslant L + 2\nu\kappa + h$. By $a \in \mathbb{R}_+$ and $c \in \mathbb{R}_+$ we abbreviate the following expressions:

$$a := a(h) := \log \frac{2\nu\kappa + h}{2\nu\kappa} \tag{B.23}$$

and

$$c := c(h) := (2\nu\kappa + h)/h. \tag{B.24}$$

In the trivial case where $\widetilde{\Pi} = \Pi = \{\widetilde{z}\}$ (i.e., $\xi(\cdot)$ is a "single peak" potential), we have that the principal eigenvalue $\lambda_1$ of the Hamiltonian $\mathcal{H}$ coincides with $\widetilde{\lambda}_1$ and

$$\big|\psi(x; \lambda_1)\big| \leqslant c(h) \exp\big\{ - A(\widetilde{\lambda}_1)|x - \widetilde{z}|\big\} \quad (x \in V)$$

according to Theorem B.1 and (B.11).

In the case of $|\widetilde{\Pi}| \geqslant 2$ or $|\Pi| \geqslant 2$, for $K \in \mathbb{N}$ and $0 < \delta < 1$ we introduce the following conditions on the sample $\{\xi(x) : x \in V\}$:

$$|\widetilde{\Pi}| \geqslant K, \tag{B.25}$$

$$\min_{u \in V \setminus \widetilde{\Pi}} \big(\widetilde{\lambda}_{K+1} - \xi(u)\big) \geqslant 2\nu\kappa^2/h, \tag{B.26}$$

$$16c(h)^2 \sum_{x \in V \setminus \{0\}} \exp\big\{ - 2(1-\delta) A(\widetilde{\lambda}_{K+1})|x|\big\} < 1, \tag{B.27}$$

$$\min_{1 \leqslant l \leqslant K} (\widetilde{\lambda}_l - \widetilde{\lambda}_{l+1}) \geqslant \mathrm{e}^{-\delta a(h) r/2} \tag{B.28}$$

and, finally,

$$r \geqslant \frac{r_0(\kappa, \nu)}{\delta(h^2 \wedge 1)} \log |\Pi|; \tag{B.29}$$

here, remember, $r$ stands for the minimum distance between sites in $\Pi$ defined by (B.5); $r_0(\kappa, \nu) > 0$ denotes a large constant which depends on $\kappa$ and $\nu$ (not on $V, h$, etc.) and which is actually specified in the proof of Theorem B.3 below.

We now associate the sites $\widetilde{z}_k \in \Pi$ with the variational series (B.22) by $\widetilde{\lambda}(\widetilde{z}_k) := \widetilde{\lambda}_k$ $(1 \leqslant k \leqslant K)$. Recall also that $\lambda_k$ and $\psi(\cdot; \lambda_k)$ denote, respectively, the $k$th eigenvalue and the corresponding (normalized) eigenfunction of the Hamiltonian $\mathcal{H}$.

**Theorem B.3** *Fix $V$ and constants $0 < L < \infty$, $1 \leqslant K < |V|$, $0 < h < \infty$ and $0 < \delta < 1/2$ which all may depend on $V$. Assume that $\{\xi(x) : x \in V\}$ satisfies (B.25)–(B.29). Then we have that, for any $1 \leqslant l \leqslant K$,*

$$|\lambda_l - \widetilde{\lambda}_l| \leqslant \exp\big\{ - 2(1-\delta) A(\widetilde{\lambda}_l) r\big\} \tag{B.30}$$

*and*

$$\big|\psi(x; \lambda_l)\big| \leqslant 4c(h) \exp\big\{ - (1-\delta) A(\lambda_l)|x - \widetilde{z}_l|\big\} \quad (x \in V), \tag{B.31}$$

*where $c(h)$ is given by (B.24).*

*Remark B.4* (i) Assumptions (B.28) and (B.29) are to avoid the interaction among single high peaks of $\xi(\cdot)$ in the model. Assumptions (B.26) and (B.27) guarantee that the interaction between a single high peak and a multiple (double, triple, etc.) one is negligible. According to Theorem B.3, the analysis of $\mathrm{Spect}(\mathcal{H}) \cap (L + 2\nu\kappa + h, \infty)$ is reduced to the study of the principal eigenvalues of the separate "single peak" Hamiltonians $\widetilde{\mathcal{H}}^{(z)}$ $(z \in \widetilde{\Pi})$.

(ii) In fact, we shall show that each eigenvalue $\lambda(z)$ of $\mathcal{H}$, which is situated in the neighborhood of $\widetilde{\lambda}(z)$, satisfies dispersion equation (B.7) and the corresponding eigenfunction is given by (B.8). Therefore, using the cluster expansion formulas for Green's function $\mathcal{G}^{(z)}(\lambda; \cdot, \cdot)$ (see Theorem B.1(iii) and its proof), we obtain the explicit estimates for the eigenpairs $\lambda(z)$, $\psi(\cdot; \lambda(z))$, provided that $\lambda(z) \geqslant L + 2\nu\kappa + h$.

*Remark B.5* For $u \in \widetilde{\Pi}$ chosen arbitrarily, $\widetilde{\lambda}(u)$ is the principal eigenvalue of $\widetilde{\mathcal{H}}^{(u)}$ if and only if $\widetilde{\lambda}(u)$ is the maximal solution of the equation

$$\widetilde{\mathcal{G}}(\lambda; u, u) = 1/\xi(u) \tag{B.32}$$

(Theorem B.1(ii)). Moreover,

$$\min_{\substack{\mu \in \mathrm{Spect}(\widetilde{\mathcal{H}}^{(u)}) \\ \mu \neq \widetilde{\lambda}(u)}} \big(\widetilde{\lambda}(u) - \mu\big) \geqslant h.$$

*Proof of Theorem B.3* We treat the case $K \geqslant 2$. For $K = 1$, the proof is similar.

For a brevity of notation, we write $\Pi^K := \{\widetilde{z}_k : 1 \leqslant k \leqslant K\}$. By $q$ we denote the right-hand side of (B.28), i.e.,

$$q := \mathrm{e}^{-\delta a r/2}. \tag{B.33}$$

We introduce the following intervals

$$I(u) := \big[\widetilde{\lambda}(u) - q/3, \widetilde{\lambda}(u) + q/3\big] \quad (u \in \Pi^K)$$

and

$$I := [\lambda_0, \infty) \quad \text{where } \lambda_0 := \widetilde{\lambda}_K - q/3. \tag{B.34}$$

Thus, by (B.28) and (B.33) we have that

$$I(u) \subset I \quad \text{and} \quad I(u) \cap I(v) = \varnothing \quad (u \in \Pi^K, v \in \Pi^K \backslash \{u\}). \tag{B.35}$$

Also, by the definition,

$$\widetilde{\lambda}(u) > \lambda_0 > \widetilde{\lambda}_{K+1} \geqslant L + 2\nu\kappa + h \quad (u \in \Pi^K). \tag{B.36}$$

For each $z \in \Pi^K$, let the subset $\Lambda(z) := \Lambda(z; \lambda_0; \delta; \xi(\cdot)) \subset I$ be defined by (B.6) with $\lambda_0$ as in (B.34). Also, by $\widetilde{q}(z)$ we denote the right-hand side of (B.30) with $\widetilde{\lambda}(z)$ instead of $\widetilde{\lambda}_l$, viz.

$$\widetilde{q}(z) := \exp\big\{ -2(1-\delta)A(\widetilde{\lambda}(z))r \big\} \quad (z \in \Pi^K). \tag{B.37}$$

**Lemma B.6** *Under the conditions of Theorem B.3 we have the following assertions.*

(j) $I(z) \subset \Lambda(z)$ *for any* $z \in \Pi^K$.
(jj) $I \backslash \bigcup_{z \in \Pi^K} I(z) \subset \bigcap_{z \in \Pi^K} \Lambda(z)$.

(jjj) *If* $z \in \Pi^K$ *and if* $\lambda \in \Lambda(z)$ *satisfies equation* (B.7), *then*

$$\big|\lambda - \widetilde{\lambda}(z)\big| \leqslant \widetilde{q}(z) \tag{B.38}$$

*and*, *consequently*, $\lambda \in I(z)$.

*Proof* (j)–(jj) Fix $z \in \Pi^K$, and rewrite $\Lambda(z)$ (B.6) as follows:

$$\Lambda(z) = \bigcap_{u \in \Pi \setminus \{z\}} \Lambda^u,$$

where

$$\Lambda^u := \left\{\lambda \geqslant \lambda_0 : \left|\frac{1}{\xi(u)} - \widetilde{\mathcal{G}}(\lambda; u, u)\right| \lambda^2 \geqslant 2b(\lambda)|\Pi|\,\mathrm{e}^{-\delta A(\lambda) r}\right\}. \tag{B.39}$$

For each $\lambda \geqslant \lambda_0$ we have that

$$2b(\lambda)|\Pi|\,\mathrm{e}^{-\delta A(\lambda) r} = \frac{4\nu\kappa(\lambda - L)}{\lambda - L - 2\nu\kappa}|\Pi|\,\mathrm{e}^{-A(\lambda)(\delta r - 1)} \leqslant \frac{q}{6} \tag{B.40}$$

by (B.36) and (B.29). On the other hand, using the properties of $\widetilde{\lambda}(u)$ ($u \in \widetilde{\Pi}$) (see Remark B.5), we obtain that, for any $\lambda \geqslant \lambda_0$ and any $u \in \widetilde{\Pi}$,

$$\begin{aligned} \left|\frac{1}{\xi(u)} - \widetilde{\mathcal{G}}(\lambda; u, u)\right| \lambda^2 &= \big|\widetilde{\mathcal{G}}(\widetilde{\lambda}(u); u, u) - \widetilde{\mathcal{G}}(\lambda; u, u)\big|\lambda^2 \\ &\geqslant \frac{|\lambda - \widetilde{\lambda}(u)|\lambda}{\widetilde{\lambda}(u)} \geqslant \begin{cases} \frac{1}{2}|\lambda - \widetilde{\lambda}(u)| & \text{if } \lambda \geqslant \frac{1}{2}\widetilde{\lambda}(u), \\ \frac{1}{2}\lambda & \text{if } \lambda < \frac{1}{2}\widetilde{\lambda}(u). \end{cases} \end{aligned} \tag{B.41}$$

Estimates (B.40) and (B.41) imply that

$$\Lambda^u \supset \big\{\lambda \geqslant \lambda_0 :\ |\lambda - \widetilde{\lambda}(u)| \geqslant q/3\big\} \quad (u \in \widetilde{\Pi}). \tag{B.42}$$

Let us consider $\Lambda^u$ (B.39) for $u \in \Pi \setminus \widetilde{\Pi}$. Since $\lambda_0 > L + 2\nu\kappa + h$, we first have that, for any $\lambda \geqslant \lambda_0$,

$$\widetilde{\mathcal{G}}(\lambda; u, u) \leqslant 1/\lambda + \sigma/\lambda^2, \tag{B.43}$$

where

$$\begin{aligned} \sigma &:= \kappa \sum_{k \in \mathbb{N}} \exp\big\{-(2k-1)A(L + 2\nu\kappa + h)\big\} \\ &= \frac{2\nu\kappa^2(2\nu\kappa + h)}{(4\nu\kappa + h)h} \leqslant \frac{2\nu\kappa^2}{h} - q \end{aligned} \tag{B.44}$$

because of assumption (B.29). Further, since $\lambda_0 \geqslant \xi(u)$, we obtain that, for any $\lambda \geqslant \lambda_0$,

$$\widetilde{\mathcal{G}}(\lambda; u, u)\lambda^2/\xi(u) \geqslant 1$$

and by (B.43)

$$\frac{1}{\widetilde{\mathcal{G}}(\lambda; u, u)} - \xi(u) \geqslant \frac{\lambda^2}{\lambda + \sigma} - \xi(u) \geqslant \lambda - \sigma - \xi(u) \geqslant \widetilde{\lambda}_{K+1} - \sigma - \xi(u) \geqslant q$$

by combining (B.44) and assumption (B.26). These estimates imply that

$$\left(\frac{1}{\xi(u)} - \widetilde{\mathcal{G}}(\lambda; u, u)\right)\lambda^2 \geqslant q \quad (\lambda \geqslant \lambda_0). \tag{B.45}$$

Now, (B.40) and (B.45) yield that $\Lambda^u \supset [\lambda_0, \infty)$, where $u \in \Pi \backslash \widetilde{\Pi}$ is chosen arbitrarily. This and (B.42) immediately imply assertions (j) and (jj).

(jjj) Let us show (B.38) by assuming that $\lambda$ belongs to $\Lambda(z)$ and satisfies equation (B.7). From (B.7) and the properties of $\widetilde{\lambda}(z)$ (see Remark B.5) we get that

$$\begin{aligned} &\left|\mathcal{G}^{(z)}(\lambda; z, z) - \widetilde{\mathcal{G}}(\lambda; z, z)\right| \\ &\quad = \left|\widetilde{\mathcal{G}}(\widetilde{\lambda}(z); z, z) - \widetilde{\mathcal{G}}(\lambda; z, z)\right| \\ &\quad \geqslant \frac{|\lambda - \widetilde{\lambda}(z)|}{\lambda\widetilde{\lambda}(z)} \geqslant \begin{cases} \frac{1}{2}|\widetilde{\lambda}(z) - \lambda|\lambda^{-2} & \text{if } \lambda \geqslant \frac{1}{2}\widetilde{\lambda}(z), \\ \frac{1}{2}\lambda^{-1} & \text{if } \lambda < \frac{1}{2}\widetilde{\lambda}(z). \end{cases} \end{aligned} \tag{B.46}$$

On the other hand, since $\lambda \in \Lambda(z)$, from (B.10) we have that the left-hand side of (B.46) does not exceed

$$b(\lambda)\lambda^{-2} \exp\bigl\{-(2-\delta)A(\lambda)r\bigr\} \leqslant \frac{1}{2\lambda^2} \exp\left\{-\left(2 - \frac{3\delta}{2}\right)A(\lambda)r\right\}$$

by (B.29). These estimates yield that

$$\left|\widetilde{\lambda}(z) - \lambda\right| \leqslant \exp\left\{-\left(2 - \frac{3\delta}{2}\right)A(\lambda)r\right\}$$

($0 < \delta < 1/2$) which in turn implies $\widetilde{\lambda}(z) - \lambda \leqslant \mathrm{e}^{-ar}$, where $a$ is given by (B.23). Consequently, $|\widetilde{\lambda}(z) - \lambda| \leqslant \widetilde{q}(z)q_1q_2$, where $\widetilde{q}(z)$ is given by (B.37) and

$$q_1 := \left(\frac{\widetilde{\lambda}(z) - L - \mathrm{e}^{-ar}}{\widetilde{\lambda}(z) - L}\right)^{-2(1-\delta)r}, \quad \text{and} \quad q_2 := \exp\{-\delta A(\lambda)r/2\}.$$

We need to estimate $q_1q_2$. In view of (B.29), we have that

$$q_1 \leqslant (1 - \exp\{-ar/2\})^{-2r} = \left[(1 - \exp\{-ar/2\})^{-ar/2}\right]^{4/a} \leqslant \mathrm{e}^{4/a}$$

according to the inequality $(1 - \mathrm{e}^{-t})^{-t} \leqslant \mathrm{e}$ for all $t \geqslant 1$. Thus, by (B.29) we have that $q_1q_2 < 1$ and, therefore, (B.38) follows. Lemma B.6 is proved. □

We now finish the proof of Theorem B.3 by using Lemma B.6 and Theorem B.1.

Fix $\lambda \in I \backslash \bigcup_{z \in \Pi^K} I(z)$. According to Lemma B.6(jj), $\lambda$ belongs to $\Lambda(z)$ for any $z \in \Pi^K$. Therefore, Lemma B.6(jjj) and Theorem B.1(ii) imply that $\lambda$ cannot be an eigenvalue of $\mathcal{H}$.

For fixed $z \in \Pi^K$, we now consider the function $\mathcal{G}^{(z)}(\lambda; z, z)$ ($\lambda \in I(z) \subset \Lambda(z)$). In view of (B.29), estimates (B.10) and (B.36) imply that $\mathcal{G}^{(z)}(\lambda; z, z)$ is bounded from above by $g_1(\lambda) := \widetilde{\mathcal{G}}(\lambda; z, z) + \frac{q}{6}\lambda^{-2}$ and from below by $g_2(\lambda) := \widetilde{\mathcal{G}}(\lambda; z, z) - \frac{q}{6}\lambda^{-2}$. Expanding $\widetilde{\mathcal{G}}(\lambda; z, z)$ in powers of $\lambda - \widetilde{\lambda}(z)$ and noting that $\widetilde{\lambda}(z)$ satisfies equation (B.32) with $u = z$, we find that there exists in $I(z)$ a solution of the equation $g_i(\lambda) = 1/\xi(z)$ for each $i = 1$ and 2. Consequently, because of the continuity of the function $\mathcal{G}^{(z)}(\cdot\,; z, z)$ in $I(z)$ (see Theorem B.1(i)), there exists in $I(z)$ a solution of the equation $\mathcal{G}^{(z)}(\lambda; z, z) = 1/\xi(z)$ which we denote by $\lambda(z)$. Now, from Theorem B.1(ii) and Lemma B.6(jjj) we note that $\lambda(z)$ is an eigenvalue of $\mathcal{H}$ such that $|\lambda(z) - \widetilde{\lambda}(z)| \leqslant \widetilde{q}(z)$. I.e., (B.30) is proved. Moreover, the eigenfunction corresponding to $\lambda(z)$ can be estimated as follows:

$$
\begin{aligned}
|\psi(x; \lambda(z))| &\leqslant |\mathcal{G}^{(z)}(\lambda(z); x, z)|/\mathcal{G}^{(z)}(\lambda(z); z, z) \\
&\leqslant 2c\xi(z)\lambda(z)^{-1}\exp\{-(1-\delta)A(\lambda(z))|x - z|\} \quad (x \in V)
\end{aligned}
$$

by (B.7) and (B.9). Here

$$
\lambda(z) \geqslant \widetilde{\lambda}(z) - \widetilde{q}(z) \geqslant \widetilde{\lambda}(z)/2 \geqslant \xi(z)/2, \tag{B.47}
$$

where the second inequality follows from (B.29). These estimates imply (B.31).

It only remains to show the uniqueness of the eigenvalue $\lambda(z)$ in $I(z)$. Let $\lambda'$ belong to $I(z)$ and satisfy (B.7). According to the resolvent identity, we may therefore write

$$
\begin{aligned}
0 &= \mathcal{G}^{(z)}(\lambda(z); z, z) - \mathcal{G}^{(z)}(\lambda'; z, z) \\
&= (\lambda' - \lambda(z)) \sum_{x \in V} \mathcal{G}^{(z)}(\lambda'; x, z)\mathcal{G}^{(z)}(\lambda(z); x, z).
\end{aligned} \tag{B.48}
$$

All together, (B.7), (B.9), (B.36) and (B.47), imply that the sum on the right of (B.48) is bigger than

$$
\frac{1}{\xi(z)^2} - \frac{16c^2}{\xi(z)^2} \sum_{x \neq 0} \exp\bigl\{-2(1-\delta)A(\widetilde{\lambda}_{K+1})|x|\bigr\} > 0
$$

by assumption (B.27). In view of (B.35), this implies that $\lambda' = \lambda(z)$. Theorem B.3 is proved. □

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